Math 614, Fall 2015 **Problem Set #1**
Due: Friday, October 2

1. Let X be an indeterminate over the integers, so that $R = \mathbb{Z}[X]$ is a polynomial ring. Let A denote the subring of R generated by $x^2 - 3$ and $2x$, and let I denote the ideal of R generated by $x^2 - 3$ and $2x$.

(a) Describe the ring R/I as an abelian group: in particular, what is its cardinality?

(b) Let $G = R/A$ (which has the structure of an abelian group but not of a ring: note that $\mathbb{Z} \subseteq A$). Show that G is isomorphic to the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$, and give a minimal set of generators.

2. Let $p_1, \ldots, p_n, \ldots$ be an infinite strictly increasing sequence of positive prime integers. Let $K_n = \mathbb{Z}/p_n\mathbb{Z}$ for $n \geq 1$ and let $K_0 = \mathbb{Q}$, the rational numbers. Let R denote the subring of the product ring $\prod_{i=0}^{\infty} K_i$ consisting of sequences $a_0, a_1, \ldots, a_n, \ldots$ such that $a_n \in K_n$ for all n, and such that for all sufficiently large n, a_n is the image of a_0 in K_n (this makes sense because if $a_0 = r/s$, $r \in \mathbb{Z}$, $s \in \mathbb{Z} - \{0\}$, s is not divisible by p_n when n is sufficiently large). Determine $\mathrm{Spec}\,(R)$, including its topology.

3. Let X be a topological space and let x be a point of X. Let S be the set of real-valued conbtinuous functions defined on an open neighborhood of x, and define f and g in S to be equivalent if their restrictions to a sufficiently small open neighborhood of x agree. The equivalence classes form a ring (you may assume this) called the *ring of germs of continuous functions at x*. Denote this ring $T = \mathbb{C}_{\mathbb{R}}(X)_x$.

(a) Show that T has a unique maximal ideal m, and determine the residue class field T/m.

(b) Show that the ring of germs at a point of $\mathbb{R}^n$ for $n \geq 1$ is not Noetherian.

4. Consider a family of sets in Spec (R) each of which is either closed or of the form $D(f)$. Suppose that every finite subfamily has nonempty intersection. Show that this family of sets has nonempty intersection.

5. Let P and Q be prime ideals of a ring R. Show that if there is no prime ideal contained in both P and Q, then P and Q have disjoint open neighborhoods. Deduce that the subspace of $\mathrm{Spec}\,(R)$ consisting of minimal primes is Hausdorff.

6. Let K be a field and $R = K[X]$ a polynomial ring in one variable over K. Prove that every K-sublagebra of R is finitely generated over K. (Compare with **EC2**.)

Extra Credit 1. Consider the subring R of the polynomial ring $\mathbb{Q}[x]$ consisting of all f that map $\mathbb{Z}$ to $\mathbb{Z}$. Is this ring Noetherian? (R is larger than $\mathbb{Z}[x]$, e.g., $\frac{1}{2}x(x-1) \in R$.)

Extra Credit 2. Let S denote the polynomial ring $K[x, y]$ in two variables over a field K, and for each positive real number r, let S_r be the K-subalgebra spanned over K by all monomials $x^a y^b$ such that $b/a < r$. Note that the S_r form a chain of subrings of $K[x, y]$, and that if $r \neq r'$ then $S_r \neq S_{r'}$. Which, if any, of the rings S_r finitely generated over K? Prove your answer.