Math 614, Fall 2015 **Problem Set #2**
Due: Wednesday, October 21

**1.** Let $R$ be a ring and let $X_1, \ldots, X_n$ be indeterminates over $R$. Consider a system $\mathcal{L}$ of $h$ linear equations in $n$ unknowns, $r_{i1}X_1 + \cdots r_{in}X_n = a_i$, $1 \le i \le h$, with coefficients in $R$ (i.e., the $r_{ij} \in R$ and the $a_i \in R$ are fixed, and one seeks values for the $X_i$ in $R$ that satisfy the equations). For every maximal ideal $m \in R$, there is a corresponding system of linear equations over $R_m$, call it $\mathcal{L}_m$, obtained by replacing every $r_{ij}$ and every $a_I$ by its image in the local ring $R_m$. Prove that $\mathcal{L}$ has a solution in $R$ iff for all maximal ideals $m$ of $R$, the system $\mathcal{L}_m$ has a solution in $\mathcal{L}_m$. (Suggestion: show the existence of a solution is equivalent to a question about membership in a submodule of a module.)

**2.** Let $\mathcal{C}$ be a category and $X$, $Y \in \mathrm{Ob}\,(\mathcal{C})$. Show that the natural transformations between $h_X$ and $h_Y$ are in one-to-one correspondence with $\mathrm{Mor}\,(Y, X)$. (Given $g : Y \to X$, one has a map ${}^gT_Z : H_X(Z) \to H_Y(Z)$ induced by composition with $g$ on the left. Show that all natural transformations from $h_X$ to $h_Y$ arise uniquely in this way.) (It is straightforward to check that if $f : Z \to Y$ and $g : Y \to X$, the natural transformation ${}^{g\circ f}T$ is ${}^fT \circ^g T$. It follows that $h_X$ and $h_Y$ are isomorphic if and only if $X$ and $Y$ are isomorphic. This shows the uniqueness, up to isomorphism, of the object representing a covariant functor.)

**3.** Let $R$ be a subring of $S$, and suppose that there is an $R$-module map $\theta : S \to R$ (it need *not* be an algebra map) such that $\theta(1) = 1$. Note that $\theta(r) = r$ for all $r \in R$. (One says that $R$ is a direct summand of $S$ as an $R$-module.)

(a) Show that for every ideal $I$ of $R$, $IS \cap R = I$.

(b) Show that the map $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ is surjective.

(c) Prove that if $S$ is a normal domain, then $R$ is a normal domain.

**4.** Let $R = \bigoplus_{n=0}^{\infty} R_n$ be an $\mathbb{N}$-graded ring. For $d > 0$, let $R^{(d)} = \bigoplus_{n=0}^{\infty} R_{nd}$ (called the $d$th *Veronese* subring of $R$). Prove that if $R$ is a normal domain, then so is $R^{(d)}$.

**5.** If $P$ is a prime ideal of $R$, $P^{(n)}$ denotes the contraction of $P^nR_P$ to $R$, and is called the $n$th *symbolic power* of $P$. Let $T = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a polynomial ring over a field $K$, where $n \ge 2$, and let $f = \sum_{i=1}^{n} x_iy_i$. Let $R = T/fT$. Let $P = (x_1, \ldots, x_n, y_1, \ldots, y_{n-1})R \subseteq R$. Show that $P$ is prime, and that $P^{(2)} \neq P^2$.

**6.** Let $R$ be a reduced ring of Krull dimension 0.

(a) Prove that for every ideal $I$ of $R$, $I = I^2$.

(b) Prove that every principal ideal of $R$ is generated by an idempotent.

**Extra Credit 3.** Let $R \subseteq S$ be rings and $s \in S$. Suppose that for every minimal prime $P$ of $S$, the image of $s$ in $S/P$ is integral over $R/(P \cap R)$ (which injects into $S/P$). Prove that $s$ is integral over $R$.

**Extra Credit 4.** Let $R$ be a ring such that $\mathrm{Spec}\,(R)$ is the union of two nonempty open sets $U$, $V$ (both will then be closed as well), i..e., $\mathrm{Spec}\,(R)$ is disconnected. Show that there are non-trivial complementary idempotents $e$, $f = 1 - e$ in $R$ such that $U = D(e)$ and $V = D(f)$. That is, $R$ is a product ring in a non-trivial way iff $\mathrm{Spec}\,(R)$ is disconnected.