Math 614, Fall 2015 **Problem Set #3**
Due: Monday, November 9

1. Let $T = K[x_1, \dots, x_n]$, $n \geq 2$, be a polynomial ring over a field K, and let f denote the sum of the square-free products of the variables taken $n-1$ at a time. Let $R = T[1/f]$. Explicitly express R as a module-finite extension of a polynomial ring over K. In particular, give the algebraically independent generators of the polynomial ring explicitly.

2. Let K be a Noetherian ring and let $R = K[u_1, \dots, u_n]$ be a finitely generated extension ring. Let $G = \{g_1, \dots, g_d\}$ be a finite group with $|G| = d$ consisting of K-algebra automorphisms of R (every element of G fixes every element of K) and let $R^G = \{r \in R : \text{ for all } g \in G, g(r) = r\}$, the *ring of invariants* of G acting on R. Prove that R^G is a finitely generated K-algebra. For each i, $1 \leq i \leq n$, let $e_{i1}, \dots, e_{id}$ be the elementary symmetric functions of the elements $g_1(u_i), \dots, g_d(u_i)$ (note that $u_i = 1_G(u_i)$). Show that every u_i is integral over $B = K[e_{ij} : 1 \leq i \leq n, \, ,1 \leq j \leq d]$, and use that $B \subseteq R^G \subseteq R$.)

3. Let R be a ring such that for every maximal ideal m of R, the ring R_m is Noetherian. Suppose also that every element of $R - \{0\}$ is contained in only finitely many maximal ideals of R. Prove that R is Noetherian.

4. Show that if the set of ideals of R that are not finitely generated is non-empty, it has a maximal element J, and that J must be prime. [Hence, if every prime ideal of R is finitely generated, then R is Noetherian.] (Suggestion: if $fg \in J$ with $f \notin J$ and $g \notin J$, then $J :_R g = \{r \in R : rg \in J\}$ is finitely generated, and so is $J + Rg$.)

5. Let K be a field, and let $R = K[x_1, \dots, x_n]$, a polynomial ring in n variables over K. For $1 \leq j \leq n$, Let F_j be a polynomial of degree $d_j \geq 1$ whose only term of degree d_j is $x_j^{d_j}$. Prove that R is a finitely generated free module over $A = K[F_1, \dots, F_n]$. Show that every K-subalgebra of R that contains $F_1, \dots, F_n$ is finitely generated over K.

6. Let $S = K[x, y, z]$ be a polynomial ring over a field and let $R = K[xy, xz, yz] \subseteq S$ Describe explicitly the image of $\mathrm{Spec}\,(S)$ in $\mathrm{Spec}\,(R)$. Is it open? Is it closed?

Extra Credit 5. Let R be a ring in which every prime ideal is an intersection of maximal ideals. Prove that every finitely generated R-algebra has the same property.

Extra Credit 6. Let $x_1, \dots, x_n \dots$ be an infinite sequence of indeterminates over a field K. Let $T = K[x_1, \dots, x_n, \dots]$. For $n \geq 1$, let P_n be the prime ideal generated by the x_j for $j \in S_n$, where S_n is the set of n integers in the closed interval $[\binom{n}{2} + 1, \, \binom{n}{2} + n]$. Let W be the complement in T of the union of all the P_n. Let $R = W^{-1}T$.

(a) Prove that $m_n = P_n R$ is maximal in R, and that $\mathrm{MaxSpec}(R) = \{m_n : n \geq 1\}$.

(b) Let L_n be the fraction field of $K[x_k : k \notin S_n]$, let $A_n = L_n[x_j : j \in S_n]$, and let $\mu_n = (x_j : j \in S_n)A_n$, which is maximal in A_n. Show that $R_{m_n} \cong A_{\mu_n}$.

(c) Show that R_{m_n} is Noetherian of Krull dimension n.

(d) Show that every element of $R - \{0\}$ is contained in only finitely many maximal ideals.

(e) Prove that R is a Noetherian domain of infinite Krull dimension.