Math 614, Fall 2015 **Problem Set #5**
Due: Wednesday, December 9

**1.** Let $\mu_1, \dots, \mu_h$ be monomials in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field $K$, where $\mu_i \not| \mu_j$ for $i \neq j$. Show that $I = (\mu_1, \dots, \mu_h)R$ is primary iff for every $x_j$ dividing some $\mu_i$, some $x_j^t \in I$.

**2.** Find an irredundant primary decomposition for the ideal $(x^3, xyzw, y^2z^2, zw^2, w^4)$ in the polynomial ring $K[w, x, y, z]$ over the field $K$. What are the associated primes? Which are minimal? Which primary components are unique?

**3.** Let $P \in \mathrm{Spec}\,(R)$, $R$ Noetherian and let $x, y \in P$ be nonzerodivisors. Show that $P$ is an associated prime of $xR$ iff it is an associated prime of $yR$.

**4.** (a) Let $M$, $N$ be $R$-modules with $\ell(M)$, $\ell(N) < \infty$. Show that $\ell(\mathrm{Hom}_R(M, N) < \infty$.

(b) Let $(*)\quad 0 \to A \to B \to C \to 0$ be an exact sequence of finite length $R$-modules. If $B \cong A \oplus C$, show that $(*)$ splits. [$0 \to \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(C, B) \xrightarrow{\beta} \mathrm{Hom}_R(C, C) \to N \to 0$ is exact, where $N := \mathrm{Coker}\,(\beta)$. Use a length argument to show that $N = 0$.]

**5.** Let $R$, $M$ be Noetherian where $M$ is an $R$-module and $\mathrm{Ass}\,_R(M) = \{P_1, \dots, P_n\}$.

(a) Show that if $I \subseteq R$ and $N = \mathrm{Ann}_M I \subseteq M$, then $\mathrm{Ass}\,(M) = \mathrm{Ass}\,(N) \cup Ass(M/N)$. (If $I = f_1, \dots, f_h$, the map $M \to M^h$ given by $m \mapsto (f_1m, \dots, f_hm)$ has kernel $N$.)

(b) Show that if $P_n$ is maximal in $\mathrm{Ass}\,_R(M)$, then $M_1 = \mathrm{Ann}_M P_n \subseteq M$ is a torsion-free module over $R/P_n$, and that $\mathrm{Ass}\,_R(M/M_1) \subseteq \mathrm{Ass}\,_R(M)$.

(c) Show that $M$ has a finite filtration whose factors are nonzero torsion-free modules over the various domains $R/P_i$, and that each $P_i$ must occur.

**6.** Let $(V, tV)$ be a Noetherian discrete valuation domain with fraction field $\mathcal{F} = V[1/t]$. Let $N = \mathcal{F}/V$. $N$ is a $V$-module generated by the classes $u_n$ of the elements $1/t^n$, $n \geq 1$.

(a) Show that every submodule $W \neq 0$ of $N$ is determined by which of the $u_n$ it contains, that $N$ has DCC but not ACC, and that every submodule $W \neq 0$ of $N$ contains $u_1$.

(b) Let $R = V \oplus N$, a $V$-algebra with the multiplication $(v \oplus n)(v' \oplus n') = vv' \oplus (vn' + v'n)$, so that $N^2 = 0$. (Assume this is a ring.) Show that $0 \oplus u_1$ is in every nonzero ideal of $R$.

(c) Show that $(0)$ is not a primary ideal of $R$. [Hence, $(0)$ has no primary decomposition.]

**Extra Credit 9.** Let $I \subseteq R$, an ideal. Show that $I \otimes M \to IM$ is an isomorphism iff for every exact sequence $0 \to A \to B \to R/I \to 0$, the map $A \otimes_R M \to B \otimes_R M$ is injective.

**Extra Credit 10.** (a) Let $R$ be a normal Noetherian domain in which 2 is a unit, and $a \in R - \{0\}$ be such that $aR \neq R$ is radical. Prove that $x^2 - a$ is irreducible over $\mathrm{frac}(R)$. Let $S = R[\sqrt{a}] = R + R\sqrt{a}$. Prove that $S$ is normal.

(b) Hence $S = \mathbb{R}[x, y](x^2 + y^2 - 1)$ is a normal domain. Note that $T = \mathbb{C} \otimes_{\mathbb{R}} S \cong \mathbb{C}[u, 1/u]$ is a PID, where $u = x + yi$ (and $1/u = x - iy$). Show that the maximal ideal $m = (x - 1, y)S$ of $S$ is not principal. (A generator would also generate $mT$: it suffices to show that a generator of $mT$ cannot be in $S$.) Also show that $m \oplus m \cong S \oplus S$. [This is another example of a projective that is not free.]