Math 614, Fall 2017 **Problem Set #1**

Due: Monday, October 2

1. (a) Show that for every integer $n > 0$, x^{2n-1}, $x^{2n} + x$, and x^{2n+1} generate $K[x]$ (the polynomial ring in one variable over the field K).

(b) Consider the subring $T = K[x^2,\, x^3 + x] \subseteq K[x]$. Let $A = K[x^2]$. Show that every element of T can be written uniquely in the form $f + (x^3 + x)g$, where $f,\, g \in A$. Conclude that every element of T of odd degree has degree at least 3. Hence, $x \notin T$ and $T \neq K[x]$.

(c) Show that x is in the field of fractions of T .

2. Let R be a commutative ring and assume that $u \in R - \{0\}$ is in every nonzero ideal of R. Let I be the annihilator of u, i.e., $I = \{r \in R : ru = 0\}$. Prove that I is maximal, and that if $f \in R - I$, then f is a nonzerodivisor on R, i.e., that if $fz = 0$ then $z = 0$.

3. Let X be a compact (i.e., quasicompact Hausdorff) space. You may assume that such a space is normal (i.e., T_4), so that disjoint closed sets have disjoint open neighborhoods. Hence, a continuous $\mathbb{R}$-valued function on a closed set $Z \subseteq X$ extends continuously to all of X (the Tietze extension theorem). Let $\mathcal{C}(X) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$.

(a) Prove that there is a bijection θ between the maximal ideals of $\mathcal{C}(X)$ and the points of X, where the maximal ideal m_x corresponding to $x \in X$ is $\{f \in \mathcal{C}(X) : f(x) = 0\}$.

(b) Prove that if we give $Y = \mathrm{MaxSpec}\big(\mathcal{C}(X)\big)$, the set of all maximal ideals of $\mathcal{C}(X)$, in the inherited Zariski topology, then $\theta : x \mapsto m_x$ is a homeomorphism of X with Y.

4. Let P and Q be prime ideals of a ring R. Show that if there is no prime ideal contained in both P and Q, then P and Q have disjoint open neighborhoods. Deduce that the subspace of $\mathrm{Spec}\,(R)$ consisting of minimal primes is Hausdorff.

5. (a) Let $R \subseteq S$ be rings and $P \in \mathrm{Spec}\, R$. Show that there exists a prime Q of S whose contraction to R is P if and only if the map $R \to R/P$ extends to a map $S \to D$, where $R/P \subseteq D$ and D is an integral domain.

(b) Let $S = K[x,\, y,\, z]$ be the polynomial ring in three variables over a field K, and $R = K[xy,\, yz,\, zx] \subseteq S$. Is $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ surjective? If not, give an explicit prime not in the image, and describe the image, if possible, as the union of an open set and a closed set.

6. Let $\mathcal{C}$ be a category, and X, Y be objects of $\mathcal{C}$. A morphism $f : X \to Y$ is called an *epimorphism* if for all objects Z and $g, h : Y \to Z$, whenever $g \circ f = h \circ f$, then $g = h$. Let $f : R \to S$ be a ring homomorphism, and suppose that every element of S has the form $f(r)/f(u)$, where $r,\, u \in R$ and $f(u)$ is invertible in S. Prove that f is an epimorphism in the category of rings.

Extra Credit 1. Consider the subring R of the polynomial ring $\mathbb{Q}[x]$ consisting of all f that map $\mathbb{Z}$ to $\mathbb{Z}$. Is this ring Noetherian? (R is larger than $\mathbb{Z}[x]$, e.g., $\frac{1}{2}x(x-1) \in R$.)

Extra Credit 2. Let $R \subseteq S$ be rings and let $s \in S$. Suppose that for every minimal prime Q of S, there is a monic polynomial f_Q in the polynomial ring $R[x]$ such that $f_Q(s) \in Q$. Show that there is a monic polynomial $f \in R[x]$ such that $f(s) = 0$.