Math 614, Fall 2017 **Problem Set #2**
Due: Wednesday, October 18

1. Let k be a positive integer such that $p = 4k+1$ is prime. Show that the ring $\mathbb{Z}[\sqrt{p}]$ is not normal, and show that its normalization consists of all elements of the form $\mathbb{Z} + \mathbb{Z}s$ for some element s in its fraction field. Give s explicitly.

2. Let R be a subring of S, and suppose that there is an R-module map $\theta : S \to R$ (it need *not* be an algebra map) such that $\theta(1) = 1$. Note that $\theta(r) = r$ for all $r \in R$. (One says that R is a *direct summand* of S as an R-module.)

(a) Show that for every ideal I of R, $IS \cap R = I$.

(b) Show that the map $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ is surjective.

(c) Prove that if S is a normal domain, then R is a normal domain.

3. Let $R = \bigoplus_{n=0}^{\infty} R_n$ be an $\mathbb{N}$-graded ring. For $d > 0$, let $R^{(d)} = \bigoplus_{n=0}^{\infty} R_{nd}$ (called the d th *Veronese* subring of R). Prove that if R is a normal domain, then so is $R^{(d)}$.

4. If P is a prime ideal of R, $P^{(n)}$ denotes the contraction of $P^n R_P$ to R, and is called the n th *symbolic power* of P. Let $T = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be a polynomial ring over a field K, where $n \geq 2$, and let $f = \sum_{i=1}^{n} x_i y_i$. Let $R = T/fT$. Let $P = (x_1, \dots, x_n, y_1, \dots, y_{n-1})R \subseteq R$. Show that P is prime, and that $P^{(2)} \neq P^2$.

5. Let R be a ring and let $S = R[x_1, \dots, x_n]$ be the polynomial ring in n variables over R. For $1 \leq i \leq n$ let $F_i \in S$ be a polynomial of degree n_i whose only term of degree n_i is $x_i^{n_i}$. Prove that $S/(F_1, \dots, F_n)S$ is module-finite over R.

6. Let R be a reduced ring of Krull dimension 0.

(a) Prove that for every ideal I of R, $I = I^2$.

(b) Prove that every principal ideal of R is generated by an idempotent.

Extra Credit 3. Let u be an element of a ring R such that $u^2 - u$ is nilpotent, i.e., the image of u is idempotent in R_{red}. Prove that there is a unique idempotent $e \in R$ such that $u = e + n$, where n is a nilpotent in R. (That is, idempotent elements u of R_{red} lift uniquely to idempotent elements e of R.)

Extra Credit 4. We have defined an integral domain R to be *normal* if it contains every element f of its fraction field that satisfies a monic polynomial with coefficients in R. Suppose that R is a domain that satisfies the weaker condition that whenever f is in the fraction field and $f^n \in R$ for some integer $n \geq 1$ then $f \in R$. Must R be normal? Prove your answer.