Math 614, Fall 2017 **Problem Set #3**
Due: Monday, November 6

**1.** Let $T = K[x_1, \ldots, x_n]$, $n \geq 2$, be a polynomial ring over a field $K$, and let $f$ denote the sum of the square-free products of the variables taken $n-1$ at a time. Let $R = T[1/f]$. Explicitly express $R$ as a module-finite extension of a polynomial ring over $K$. In particular, give the algebraically independent generators of the polynomial ring explicitly.

**2.** Let $K$ be a Noetherian ring and let $R = K[u_1, \ldots, u_n]$ be a finitely generated extension ring. Let $G = \{g_1, \ldots, g_d\}$ be a finite group with $|G| = d$ consisting of $K$-algebra automorphisms of $R$ (every element of $G$ fixes every element of $K$) and let $R^G = \{r \in R : \text{ for all } g \in G, g(r) = r\}$, the *ring of invariants* of $G$ acting on $R$. Prove that $R^G$ is a finitely generated $K$-algebra. For each $i$, $1 \leq i \leq n$, let $e_{i1}, \ldots, e_{id}$ be the elementary symmetric functions of the elements $g_1(u_i), \ldots, g_d(u_i)$ (note that $u_i = 1_G(u_i)$). Show that every $u_i$ is integral over $B = K[e_{ij} : 1 \leq i \leq n, , 1 \leq j \leq d]$, and use that $B \subseteq R^G \subseteq R$.)

**3.** Let $R$ be a ring such that for every maximal ideal $m$ of $R$, the ring $R_m$ is Noetherian. Suppose also that every element of $R - \{0\}$ is contained in only finitely many maximal ideals of $R$. Prove that $R$ is Noetherian.

**4.** Show that if the set of ideals of $R$ that are not finitely generated is non-empty, it has a maximal element $J$, and that $J$ must be prime. [Hence, if every prime ideal of $R$ is finitely generated, then $R$ is Noetherian.] (Suggestion: if $fg \in J$ with $f \notin J$ and $g \notin J$, then $J :_R g = \{r \in R : rg \in J\}$ is finitely generated, and so is $J + Rg$.)

**5.** Let $S = R[x]$, the polynomial ring in one variable over $R$. Show that the a chain of prime ideals of $S$ lying over a given prime ideal $P$ of $R$ has length at most one. Show that if $R$ has finite Krull dimension $d$, then the Krull dimension $n$ of $S$ is such that $d+1 \leq n \leq 2d+1$. (In the Noetherian case, $n = d+1$. In the general case, the statement made here is sharp.)

**5.** Let $K$ be an algebraically closed field.

(a) Let $f : K^2 \to K^2$ be the morphism such that $f(x, y) = (x, 1+xy)$ for all $x, y \in K$. Find the image of $f$, and show that it is neither open nor closed in $K^2$.

(b) Let $g : K^2 \to K^2$ be such that $g(x, y) = \big(x + y(1+xy), 1+xy\big)$ for all $x, y \in K$. Find the image $U$ of $g$, and show that it is open in $K^2$. Describe the sets $A_i \subseteq U$ of points $P$ such that $g^{-1}(P)$ has $i$ elements for $i = 1, 2$.

**Extra Credit 5.** Let $K$ be an algebraically closed field. Let $P_1, \ldots, P_n$ be any $n$ distinct points of $K^2$.

(a) Let $Q_1, \ldots, Q_n$ be any set of $n$ distinct points of $K^2$. Prove that there is an isomorphism $f : K^2 \to K^2$ such that $f(P_i) = Q_i$, $1 \leq i \leq n$.

(b) Show that there is a morphism $g : K^2 \to K^2$ whose image is precisely $K^2 - \{P_1, \ldots, P_n\}$.

**Extra Credit 6.** Let $R$ be a ring in which every prime ideal is an intersection of maximal ideals. Prove that every finitely generated $R$-algebra has the same property.