Math 614, Fall 2017 **Problem Set #4**
Due: Monday, November 27

1. Consider a system of m linear equations in n variables over a commutative ring R, $\sum_{i=1}^{n} r_{ij}x_i = r_j$, where $1 \leq j \leq m$ indexes the equations and the r_{ij}, r_j are given elements of R. Prove that the equations have a solution in R if and only if for every maximal ideal of R the corresponding system in which the coefficients are replaced by their images in R_m has a solution in R_m.

2. Let $K \subseteq L$ be fields and let $S = L[[x]]$ be the formal power series ring in one variable over L. Let $R = K + xL[[x]]$, the subring of R consisting of all power series with constant term in K. Prove that R is a Noetherian ring if and only if L is a finite algebraic extension of K. Prove that if L is algebraic over K then that R is normal if and only if $K = L$.

3. Let A be an R-module, let F and G be projective R-modules, and let $\alpha : F \twoheadrightarrow A$ and $\beta : G \to A$ be surjections. Let $M = \mathrm{Ker}\,(\alpha)$ and let $N = \mathrm{Ker}\,(\beta)$. Prove that $M \oplus G \cong N \oplus F$. (Hence, if F and G are finitely generated, then M is finitely generated if and only if N is finitely generated.)

4. Let $K \to L$ be a ring homomorphism, let U, V, and W be K-modules, let $B : U \times V \to W$ be a K-bilinear map, and let U_L, V_L and W_L denote $L \otimes_K U$, $L \otimes_K V$, $L \otimes_K W$, resp.

(a) Use class results on $\otimes$ to show that there is an L-bilinear map $B_L : U_L \times V_L \to W_L$ such that $B_L(c \otimes u, d \otimes v) = cd \otimes B(u, v)$ for all $c, d \in L$, $u \in U$, $v \in V$.

(b) Assume in addition that K is an algebraically closed field and that L is a field extension. Suppose that B has the property that its value on any pair of vectors, both nonzero, is nonzero. Prove that B_L has the same property. [Hilbert's Nullstellensatz may be useful.]

5. Let D be an integral domain that is an algebra over an algebraically closed field K.

(a) Let L be a field extension of K. Prove that $L \otimes_K D$ is a domain.

(b) Let C be any other integral domain over K. Prove that $C \otimes_K D$ is an integral domain.

6. (a) Let R be a ring with a unique maximal ideal m, i.e., a quasilocal ring. Show that R/m is flat as an R-module if and only if $m = 0$.

(b) Show that a ring R has the property that every R-module is flat if and only if R is reduced and has Krull dimension 0.

Extra Credit 7. (a) Let $K \subset L$ be a proper field extension, and let R, S be nonzero L-algebras, which we may also view as K-algebras by restriction of scalars. Show that the natural surjection $R \otimes_K S \twoheadrightarrow R \otimes_L S$ is *never* an isomorphism.

(b) Let L be a field extension of K that is not finitely generated as a field extension. Prove that $L \otimes_K L$ is not a Noetherian ring.

Extra Credit 8. Let M be a finitely generated R-module and let let $f : M \to M$ be surjective. Prove that f is an isomorphism. (You may *not* assume that R or M is Noetherian.)