1.(a) $h^2 = x^{10} + 2x^6 + x^2 = f^2g - 2f^2 = x^2$, and so $x^2 \in K[f, g, h]$. Then $x = h - x^2 g \in K[f, g, h]$, so $K[f, g, h] = K[x]$. There are many other ways to obtain x. $\square$
(b) Let $u := x^{2h+1} + x$, so that $S = A[u]$. Since $u^2 = x^{4h+2} + 2x^{2h+2} + x^2 \in A$, all even powers of u are in A and all odd powers are in Au. Hence, every polynomial in $F \in A[u]$ is in $Au + A$, and has the form $au + b$ with $a, b \in A$. if $au + b = a'u + b'$, then $(a - a')u = b - b'$, which is impossible unless one (and hence, both) sides are 0: if they are nonzero, the lefthand side has odd degree and the right hand side has even degree. We cannot write $x = au + b$ with $a, b \in A$, since u has too large a degree unless it is 0, and $x \notin K[x^2]$. We give a vector basis for $K[x]$ over S. Since all even powers of x are in S, the quotient is spanned by the odd powers of x. We show that the h images of the odd powers $\{x^{2t-1} : 1 \le t \le h\}$ are a basis for $K[x]/S$. First note that these elements are linearly independent over K: if a nonzero K-linear it combination of them were in S, we could write it as $au + b$ with $a, b \in A$. The top degree term in au is of odd degree and at least $2h + 1$: it cannot be canceled by an element of A, which forces $a = 0$, while b has even degree. Note that $x^{2h+1} \cong -x \bmod S$. Since $x^{2j}(x^{2h+1} + x \in S$, we have $x^{2(h+j)+1} \cong -x^{2j+1}$ for $i \ge 0$, and it follows by induction on j that $x^{2h+2j+1} \cong \pm x^{2t-1}$ for $1 \le t \le h$: each higher odd power is congruent, up to sign, to a lower odd power until the exponent drops to $2h - 1$ or a smaller number. Thus, dim $(K[x]/S) = h$.
(Since $[K(x) : K(x^2)] = 2$, there is no field strictly between, and so $K(x^2)[u] = K(x)$.)

2. (a) If there are two initial objects there is a unique morphism between them in each direction. The compositions must be the respective identity morphisms: since there is only one morphism from an initial object to itself, it is the identity.

(b) $\mathbb{Z}$ is an initial object, and 0 is a terminal object.

(c) The empty set $\emptyset$ is an initial object, while any set with one element is a terminal object.

3. If h vanishes on Y, let $q(x) = h(x)/f(x)$ for $x \notin Y$ and $h(x) = 0$ if $x \in Y$. Then $h = qf$, while it is clear that every multiple of f vanishes on Y. This proves (a).

Clearly, any function of the form $pf + qg$ vanishes on $Y \cap Z$. If h vanishes on $Y \cap Z$, choose $p(x)$ to be $h(x)/f(x)$ and $q(x)$ to be 0 on $X - Y$, choose $p(x)$ to be 0 and $q(x)$ to be $h(x)/g(x)$ on $Y - (Y \cap Z)$, and choose the value 0 for both $p(x)$ and $q(x)$ on $Y \cap Z$. Then $h = pf + qg$ everywhere. $\square$

4. If a minimal prime P does not contain the image of x_t, it must contain all of the images of the x_s for $s \ne t$, since $x_s x_t = 0 \in P$. But the ideal P_t generated by the images of all the x_s for $s \ne t$ is prime, with quotient $B/P_t \cong K[x_t]$, a polynomial ring. Hence, P_t is a minimal prime. Every minimal prime must fail to contain the image of some x_t, for the ideal generated by the images of all the x_t is prime, but not minimal, since any of the P_t is strictly smaller. Any prime that is not minimal must contain one of the P_t, and so corresponds to a nonzero prime of $K[x_t]$. The only primes in $K[x_t]$ are (0) are the maximal ideals, each of which is generated by a (unique) monic irreducible polynomial in x_t of positive degree. Thus, every non-minimal prime is maximal, and is generated by all the variables but one, say x_t, and a monic irreducible polynomial of degree > 0 in x_t. $\square$

5. Let $W = \{w_1 \cdots w_n : w_i \in W_i, 1 \le i \le n\}$. W is the smallest multiplicative system that contains all the W_i (it does contain them all, since each W_j for $j \ne 1$ contains 1). If

P is a prime disjoint from all the W_i, the $R - P$ is a multiplicative system that contains all the W_i. Hence, we cannot have $0 \in W$. But if $0 \notin W$, a class theorem asserts the existence of prime P containing 0 and disjoint for W, and this P will be disjoint from $\bigcup_{i=1}^{n} W_i \subseteq W$. $\square$

6. Let $W_\lambda = \{a_\lambda^s : s \in \mathbb{N}\}$ be the multiplicative system generated by a_λ. The hypothesis that each $U_{\lambda_1} \cap \cdots \cap U_{\lambda_n}$ is nonempty is equivalent to the assertion that $W_{\lambda_1} \cdots W_{\lambda_n}$ does not contain 0. Let W be the union of these: it is a directed union of multiplicative systems, which implies that it is a multiplicative system. It does not contain 0. Hence, there is a prime ideal disjoint from the union of these multiplicative systems, and this prime is in the intersection of all the U_λ. $\square$

EC1. To check that R injects into S, it suffices to show that $x + yz, y + xz$, and z are algebraically independent, or that $K(x + yz, y + xz, z) \subseteq K(x, y, z)$ is an algebraic extension. Throughout the rest of the discussion, we identify w with z. Since $x + zy = u$ and $zx + y = v$, $(1 - z^2)x = u - zv$, and $x = (u - zv)/(1 - z^2) \in K(u, v, z)$. Similarly, $y = (v - zu)/(1 - u^2) \in K(u, v, z)$, so the fraction fields are actually the same. In fact, this shows that $K[u, v, z]_{1-z^2} = K[x, y, z]_{1-z^2}$, which implies that every prime that does not contain $1 - z^2$ in R is in the image of f^*. This shows that the image of f^* contains a dense open subset of Spec (R), namely $D_R(1 - z^2)$ (this is non-empty and open, and it must be dense since R is a domain and Spec (R) is therefore irreducible).
If a prime P of R contains $1 - z^2$, it contains $1 - z$ or $1 + z$. Thus, a prime Q that lies over P must contain $1 - z$ or $1 + z$ in S. We first analyze the image of the primes in S containing $1 - z$: it is the image of $V_S(1 - z) \approx \mathrm{Spec}\big(S/(1 - z)S\big) \cong \mathrm{Spec}\,(K[x, y])$ in $V_R(1 - z) \approx \mathrm{Spec}\big(R/(1 - z)R \approx \mathrm{Spec}\,(K[u, v])\big)$ under the map $g^* = \mathrm{Spec}\,(g)$, where $g : K[u, v] \to K[x, y]$ K-algebras such that $u \mapsto x + y$ and $v \mapsto x + y$. g factors $K[u, v] \xrightarrow{\alpha} K[x + y] \xrightarrow{\iota} K[x, y]$, where ι is the inclusion map. Here, the map of spectra $\iota^* : \mathrm{Spec}\, K[x, y] \to \mathrm{Spec}\,(K[x + y])$ is surjective because, for example, ι has a left inverse η over K such that $\eta : x \mapsto x + y$ and $\eta : y \mapsto 0$ (there are many choices η). Since $\eta\iota$ is the identity, $\iota^*\eta^*$ is the identity and ι^* is surjective. The map α has kernel $(u - v)K[u, v]$ and is surjective, so that $K[x + y] \cong K[u, v]/(u - v)K[u, v]$, and the image α^* is $V(u - v)$ in $K[u, v]$, and so this is the image of g^* as well. Since Spec $(K[u, v])$ is identified with $V_R(z - 1)$, it follows that the image of $V_S(z - 1)$ is $V_R(z - 1, u - v)$. In an entirely similar way, if we form the quotients by $(1 + z)R$ and $(1 + z)S$, we get a composite map $K[u, v] \to K[x, y] \subseteq K[x, y]$ such that $u \mapsto x - y$ and $v \mapsto -(x - y)$. The kernel is $(u + v)K[u, v]$, and an entirely similar analysis shows that the image of $V_S(1 + z)$ is $V_R(1 + z, u + v)$. Hence, $W = D(1 - z^2) \cup V(1 - z, u - v) \cup V(1 + z, u + v)$, all taken in R. W is not all of Spec (R): for example the prime $(1 - z, u, v - 1)$ is not in W. W is not closed (its closure is all of Spec (R), since $D(1 - z^2)$ is dense) and W is not open: if it were open, its intersection with $V(1 - z)$ would be open in $V(1 - z)$, and that intersection is $V(1 - z, u - v)$ (this requires a little bit of extra thought in characteristic 2, where $V(1 - z, u - v) = V(1 + z, u + v)$). In summary, f^* is not surjective, its image is neither open nor closed, but it does contain a dense open subset of Spec (R).

EC2. It is clear that $S_r \subset S_{r'}$ strictly, since if b/a is a rational number with $r' < a/b < r$ then $x^a y^b \in S_{r'} - S_r$. Fix r and suppose that there were a finite set of pairs $(a_1, b_1), \cdots (a_n, b_n)$ such that the elements $x^{a_n} y^{b_n}$ generate $\mathfrak{m}_r$. All of the b_i/a_i are $> r$. Choose b/a strictly between $\min\{b_i/a_i : 1 \le i \le n\}$ and r.

Now suppose that $\mathfrak{m}_r$ is finitely generated over K. It follows that it is generated by finitely many elements $\mu_i = x^{a_i}y^{b_i}$ where $b_i/a_i > r$. It follows by induction on the degree that the μ_i generate S_r over K. (A monomial μ of S_r of least (necessarily positive) degree not in $K[\mu_1, \ldots, \mu_n]$ can be written in as a linear combination of the μ_i, and by equating terms of the same degree as μ on both sides, one gets such a representation $\mu = \sum_{j=1}^{n} f_j \mu_j$ where the f_j have lower degree than μ or are zero. But then, the f_j are in $K[\mu_1, \ldots, \mu_n]$.) Choose a rational number $b/a > r$ but strictly smaller than all the b_i/a_i for $1 \leq i \leq n$. When we multiply two (or finitely many) monomials $x^c y^d$ with $d/c > b/a$, we get a monomial with the same property. Hence, $x^a y^b$ is not in the ring $K[\mu_1, \ldots, \mu_n]$, a contradiction. $\square$