Math 614, Fall 2020 **Problem Set #2**
Due: Monday, October 19

1. (a) Which elements in the polynomial ring $K[x,\, y,\, z]$ over the field K are integral over the subring $K[x^{13}+x^5,\, y^{17}+y^7,\, z^{23}+z^{11}]$? Explain your answer.

(b) Let S be the ring of elements in $\mathbb{Q}[\sqrt{-19}]$ integral over $\mathbb{Z}$. Show that there is an element $s \in S$ such that $S = \mathbb{Z} + \mathbb{Z}s$. Give a choice of s explicitly.

2. Let $A \subseteq S$ be rings and let $f,\, g \in S[x]$ be monic polynomials. Let R be the ring generated over A by the coefficients of the product polynomial fg. Show that if S is a domain, then every coefficient of f and of g is integral over R. [Suggestion: Enlarge S to an algebraically closed field L. Explain why all the roots of fg are integral over R. Express the coefficients of f and of g in terms of these roots. The result holds when A is not a domain, by a slightly different argument.]

3. Let K be an infinite field and let $R = K[x_1,\, \ldots\,, x_n]$ be a polynomial ring. Let F be a polynomial of degree $d \geq 1$ and let F_d be the homogeneous component of F of degree d: all terms in F_d have degree d. Let X be the $n \times 1$ column vector whose i th entry is x_i, $1 \leq i \leq n$. Show that there is an invertible matrix $A = \bigl(a_{ij}\bigr)$ with entries in K such that the K-automorphism of R that maps x_i to the i th entry of AX maps F to an essentially monic polynomial in $x_1,\, \ldots\,, x_n$. Note that it suffices to make F_d essentially monic in x_n, since the automorphism preserves degree. (This gives an alternative proof of Noether normalization over an infinite field.)

4. Let $K \subseteq L$ be fields, where K is algebraically closed. Let $R = K[x_1,\, \ldots\,, x_n] \subseteq S = L[x_1,\, \ldots\,, x_n]$ be polynomial rings.

(a) Given a finite set of polynomial equations over K, show that if they have a simultaneous solution in L, then they must have a simultaneous solution in K.

(b) Let $f \in R$ be a polynomial of degree $d \geq 1$ in n variables. Show that if f factors into the product of two polynomials of lower degree in S, then it also factors in this way in R.

5. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, and let $\underline{a}_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{N}^n$, $1 \leq i \leq h$. Let B denote the K-subalgebra of R generated by the h monomials $\mu_i = x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}$, $1 \leq i \leq h$. Prove that the Krull dimension of B is the same as the $\mathbb{Q}$-vector space dimension of the $\mathbb{Q}$-span of the vectors $\underline{a}_1,\, \ldots\,, \underline{a}_h$.

6. Let (V, tV) be a DVR with fraction field $L = V[1/t]$, and let $E = L/V$, which is a V-module. Show that the submodules of E consist of 0, E, and those of the form Vu_n, where $u_n = [1/t^n]$, $n \geq 1$ is an integer. Show that E has DCC but not ACC as a V-module.

Extra Credit 3. Let R be a domain, and let $a,\, b \in R - \{0\}$ be such that $aR \cap bR = abR$. Suppose that R_a and R_b are normal. Prove that R is normal.

Extra Credit 4. Let R be a finitely generated algebra over a field K *of Krull dimension one.* Prove that there is a positive integer N, which depends on R, such that every ideal of R is generated by at most N elements. (This is false in the polynomial ring $S = K[x, y]$: the ideal $(x^n,\, x^{n-1}y,\, \ldots\,,\, x^i y^{n-i},\, \ldots\,,\, , xy^{n-1},\, y^n)S$ needs $n+1$ generators.)