Math 614, Fall 2020 **Problem Set #3**
Due: Monday, November 2

**1.** (a) Let $M$ be a module that is not Noetherian. Show that among the submodules of $M$ that are not finitely generated, there is a maximal one.

(b) Let $R$ be a ring that is not Noetherian, and let $I$ be maximal among ideals that are not finitely generated. Show that $I$ is prime. (If $ab \in I$, $a \notin I$, $b \notin I$, one may consider $I + aR$ and $J = \{r \in R : ra \in I\}$.)

**2.** Let $R$ be any commutative ring, and let $S = R[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $R$. Let $Q_0 \subset Q_1 \subset \cdots \subset Q_k$ be a strictly ascending chain of prime ideals of $S$ all of which lie over the same prime $P$ in $R$. Show that $k \leq n$.

**3.** Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be an $\mathbb{N}$-graded algebra over a Noetherian ring $R_0$. Let $I$ be the ideal $\bigoplus_{n=1}^{\infty} R_n$. Prove that $R$ is Noetherian iff the ideal $I$ is generated by finitely many homogeneous elements $F_1, \ldots, F_m$, in which case $R = R_0[F_1, \ldots, F_m]$.

**4.** Let $K$ be a field and $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ indeterminates over $K$. Show that $K[x_i y_j : 1 \leq i \leq r, 1 \leq j \leq s] \subseteq K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ is normal.

**5.** Let $K$ be an algebraically closed field.

(a) Consider $C = \mathcal{V}(x_1^2 - x_2^3)$ in $\mathbb{A}_K^2$. Consider the map $\alpha : \mathbb{A}_K^1 \to C$ such that $\alpha(a) = (a^3, a^2)$ for all $a \in K$. Is this map a bijection of closed algebraic sets? What is the corresponding map of coordinate rings? Is $\alpha$ an isomorphism of closed algebraic sets?

(b) Assume that $K$ has positive prime characteristic $p$ and consider the map $F : \mathbb{A}_K^1 \to \mathbb{A}_K^1$ such that $F(a) = a^p$ for all $a \in K$. Is this map a bijection? What is the corresponding map of coordinate rings? Is $F$ an isomorphism of closed algebraic sets?

**6.** Let $S$ be any $K$-subalgebra of the polynomial ring $K[x]$ in one variable over a ring $K$. Prove that if $K$ is a field, then $S$ is finitely generated over $K$ and, hence, Noetherian. Is this true when $K = \mathbb{Z}$? Prove your answer.

**Extra Credit 5.** Let $R \subseteq S$ and suppose that $u \in S$ is such that its image in $S/\mathfrak{p}$ is integral over $R/(\mathfrak{p} \cap R)$ for all minimal primes $\mathfrak{p}$ in $S$. Must $u$ be integral over $R$?

**Extra Credit 6.** Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. Let $k$ be an integer with $0 \leq k < n$ and let $I_k$ be the ideal generated by all of the products $x_i x_{i+1} \cdots x_{i+k}$, where the subscripts are read modulo $n$. (E.g, if $n = 3$, $I_0 = (x_1, x_2, x_3)R$, while $I_1 = (x_1x_2, x_2x_3, x_3x_1)R$.) What are the minimal primes of $R/I_k$? What is the Krull dimension of this ring?