Math 614, Fall 2020 **Problem Set #3: Solutions**

1. (a) The union of a chain of non-finitely generated submodules is a submodule: any two elements belong to one of the modules in the chain, and, hence the union has the closure properties needed to be a submodule. If the union Q is finitely generated, each of the generators m_k is in one of the modules N_k in the chain. Then all of them would belong to the largest submodule N among the N_k, and they would generate Q. It follows that $N = Q$, is finitely generated, a contradiction. Hence, Zorn's lemma applies. $\square$

(b) If I were not prime and a, b as specified, then $I + aR$ and $I :_R a \supseteq I + bR$ strictly contain I, and have finite sets of generators , say $i_1 + r_1a_1, \ldots, r_h + r_ka_h$ and $g_1, \ldots, g_k$. Then $i_1, \ldots, i_h, g_1a, \ldots, g_ka \in I$. We claim they generate I. For if $i \in I$, then $i \in I + Ra$ and so we may write $i = u_1(i_i + ar_1) \cdots + u_h(i_h + ar_h) = (u_1r_1 + \cdots + u_hr_h) + au$, and where $u = u_1r_1 + \cdots + u_hr_h$. But then $ua \in I$, and so $u \in I :_R a$ and can be writtten as $r_1'g_1 + \cdots + r_k'g_k$, and it follows that $i \in (i_1, \ldots, i_h, g_1a, \ldots, g_ka)R$. $\square$

2. Primes of S lying over P are those that contain PS and are disjoint from $R - P$. These correspond bijectively via contraction (which is order-preserving) to the primes of the fiber $(R-P)^{-1}R[x]/PR[x_1, \ldots, x_n] \cong (R-P)^{-1}(R/P)[x_1, \ldots, x_n] \cong \kappa[x]$, where κ is the field $(R - P)^{-1}(R/P)$. But $\kappa[x_1, \ldots, x_n]$ has Krull dimension n by a class theorem. Hence a chain of primes lying over P corresponds to a chain of primes
in $\kappa[x_1, \ldots, x_n]$, and so has length at most n. $\square$

3. If R is Noetherian, I is evidently finitely generated. Each generator is a finite sum of homogeneous components, and these are homogeneous generators. Conversely, let $F_1, \ldots, F_m$ be homogeneous generators of I with respective positive degrees $d_1, \ldots, d_m$. If $G \in R$, let $[G]_k$ denote the homogeneous component of G in degree k. Since every element of R is a sum of homogeneous components, it will suffice to prove by induction on the degree of a homogeneous element $r \in R$ that $r \in R_0[F_1, \ldots, F_m]$. If the degree h of r is 0, then $r \in R_0$ and this is clear. If $h > 0$ then $h \in I$ and we may write $r = \sum_{j=0}^m F_jG_j$ with $G_j \in R$. Comparing the elements of degree h on both sides, we have that $r = \sum_{j=0}^m F_j[G_j]_{h-d_j}$. Since every $d_j \geq 1$, the degree of every $[G_j]_{h-d_j} < h$. By the induction hypothesis, every $[G_j]_{h-d_j} \in R_0[F_1, \ldots, F_m]$, and so the same is true for r.

4. By Prooposition 5.11(e) of the lecture notes, it suffices to show that $R = K[x_iy_j : 1 \leq i \leq r, 1 \leq j \leq s]$ is direct summand of $S = K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ as an R-module, since the polynomial ring S is a UFD and therefore normal. It is easy to see that R is spanned over K by all monomials $\mu\nu$ where mu is a monomial in $x_1, \ldots, x_r$, ν is a monomial in $y_1, \ldots, y_s$, and $\deg(\mu) = \deg \nu$. Let W be the span of all monomials $\mu\nu$ where mu is a monomial in $x_1, \ldots, x_r$, ν is a monomial in $y_1, \ldots, y_s$, and $\deg(\mu) \neq \deg \nu$. As a K-vector space, $S = R \oplus W$. To compete the proof, it will suffice to show that W is an R-module. But this follows from the distributive law and the fact that if $\mu\nu \in R$ and $\mu'\nu' \in W$, then their product $(\mu\mu')(\nu\nu') \in W$.

5. Since $(a^3)^2 = (a^2)^3$ the map is well-defined. Given $b, c \in K$ such that $b^2 = c^3$, note that $b = 0$ iff $c = 0$ and $(0, 0)$ is the image of 0. If $b, c \neq 0$, then $(b/c)^2 = b^2/c^2 = c^3/c^2 = c$ and $(b/c)^3 = b^3/c^3 = b^3/b^2 = b$. This shows the map is surjective. Final if $(a_1^2, a_1^3) = (a_2^2, a_2^3)$ then $a_1 = 0$ iff $a_2 = 0$ while if both are nonzero then $a_1 = a_1^3/a_1^2 = a_2^3/a_2^2 = a_2$. Thus, the map is injectve as well, and is bijective. The corresponding map of coordinate rings is

$K[x_1, x_2]/(x_1^2 - x_2^3)$ (the polynomial in the denominator is irreducible and therefore prime) $\to K[x]$ such that $x_1 \mapsto x^3$ and $x_2 \mapsto x_2$. The image is $K[x^2, x^3] \subsetneq K[x]$, a proper subring, and so, by the category anti-equivalence, the map is not an isomorphism: its inverse is not regular over K. Note that the surjection $K[x_1, x_2]/(x_1^2 - x_2^3) \twoheadrightarrow K[x^2, x^3]$ is an isomorphism: both are domains of dimension 1, and so there cannot be a kernel. $\square$

(b) The map is a bijection, since every element of K has a unique pth root. Note that $a^p = b^p$ implies that $(a-b)^p = 0$ and so $a = b$. The corresponding map of coordinate rings is $K[x] \to K[x]$ such that $x \mapsto x^p$. Since $K[x^p]$ is a proper subring of $K[x]$, the map of coordinate rings is not an isomorphism, and by the category anti-equivalence, the map of algebraic sets is not an isomorphism. $\square$

6. If the subring is simply K the result is obvious. If not, it contains a nonzero monic polynomial f, and lies between $A = K[f]$ and $K[x]$. Since x satisfies the monic polynomial $f(X) - f = 0$ over $K[f]$, $K[x]$ is module-finite over the Noetherian ring $K[f]$, and so is a Noetherian module over $K[f]$. The K-algebra S is a submodule of $K[x]$ over $K[f]$, and so module-finite over $K[f]$. Hence, it is finitely generated as an algebra over $K[f]$ as well, and this implies it is a finitely generated K-algebra. $\square$

The corresponding result is false over Z. In the subring $S = Z[2x, 2x^2, 2x^3, \cdots, 2x^n, \cdots \in \mathbb{Z}[x]$ the ideals $I_n = (2x, 2x^2, \cdots, 2x^n)S$ form a strictly increasing chain. (The image T of this ring under the composite map $S \subseteq Z[x] \to (\mathbb{Z}/4\mathbb{Z})[x]$ is $(\mathbb{Z}/4)[u_1, \ldots, u_n, \ldots]$ where u_n is the image of $2x^n$, $n \geq 1$. In T, every u_n is killed by 2, and $u_i u_j = 0$ for all i, j, so T, as an abelian group, is the direct sum of $\mathbb{Z}/4\mathbb{Z}$ and the vector space V over $\mathbb{Z}_2$ spanned by the u_i, which form a basis for V. The expansions of the ideals to T also give an infinite strictly ascending chain.) $\square$

EC5. The element u must be integral over R. The values of the monic polynomials in u with coefficients in R form a multiplicative system W. If u is not integral over R, then $0 \notin W$. Then there is a prime of S that does not meet W, and, hence, a minimal prime Q of S that does not meet W. The image of u mod Q is not integral over $R/(Q \cap R)$, a contradiction. $\square$

EC6. Any prime will have to contain at least one variable from each monomial in I_k. The minimal primes are therefore generated by subsets of the variables minimal with respect to the property that they contain at least one variable from each consecutive string of $k+1$ variables mod n. These correspond to minimal ascending sequences of indices $1 \leq i_1 < i_2 < \cdots < i_h \leq n$ (the subscripts on the variables occurring in the minimal prime) such that $i_{t+1} - i_t \leq k$ for $1 \leq t \leq h$, where $i_{h+1} = i_1 + n$: these differences give the lengths of the intervals of consecutive variables not used in the minimal prime. "Minimal" means that the condition fails if any i_t is omitted. The Krull dimension of the quotient is therefore $n - h$ where h is the smallest cardinality of a minimal ascending sequence of indices with the required property: this is the largest dimension one gets when one kills a minimal prime. We show that $h = \lceil \frac{n}{k+1} \rceil$. Since each variable occurs in $k+1$ strings of consecutive variables and there are n such strings, we must use at least $n/(k+1)$ variables to meet all the strings, and, hence, at least $h = \lceil n/(k+1) \rceil$. But this value of h works: use the $x_{t(k+1)+1}$, $0 \leq t \leq h-1$. $\square$