Math 614, Fall 2020 **Problem Set #4**
Due: Monday, November 16

1. Let A be an R-module, let F and G be projective R-modules, and let $\alpha : F \twoheadrightarrow A$ and $\beta : G \to A$ be surjections. Let $M = \mathrm{Ker}\,(\alpha)$ and let $N = \mathrm{Ker}\,(\beta)$. Prove that $M \oplus G \cong N \oplus F$. (Hence, if F and G are finitely generated, then M is finitely generated if and only if N is finitely generated.)

2. Let $K \to L$ be ring homomorphism, let U, V, and W be K-modules, let $B : U \times V \to W$ be a K-bilinear map, and let U_L, V_L and W_L denote $L \otimes_K U$, $L \otimes_K V$, $L \otimes_K W$, resp.

(a) Use class results on $\otimes$ to show that there is an L-bilinear map $B_L : U_L \times V_L \to W_L$ such that $B_L(c \otimes u, d \otimes v) = cd \otimes B(u, v)$ for all $c, d \in L$, $u \in U$, $v \in V$.

(b) Assume in addition that K is an algebraically closed field and that L is a field extension. Suppose that B has the property that its value on any pair of vectors, both nonzero, is nonzero. Prove that B_L has the same property. [Hilbert's Nullstellensatz may be useful.]

3. Let D be an integral domain that is an algebra over an algebraically closed field K.

(a) Let L be a field extension of K. Prove that $L \otimes_K D$ is a domain.

(b) Let C be any other integral domain over K. Prove that $C \otimes_K D$ is an integral domain.

4. (a) Let R be a ring with a unique maximal ideal m, i.e., a quasilocal ring. Show that R/m is flat as an R-module if and only if $m = 0$.

(b) Show that a ring R has the property that every R-module is flat if and only if R is reduced and has Krull dimension 0.

5. Let M and N be finitely generated modules over a Noetherian ring R. Suppose that there are surjections $M \twoheadrightarrow N$ and $N \twoheadrightarrow M$. Prove that these maps must be isomorphisms.

6. (a) Let $K \subset L$ be a proper field extension, and let R, S be nonzero L-algebras, which we may also view as K-algebras by restriction of scalars. Show that the natural surjection $R \otimes_K S \twoheadrightarrow R \otimes_L S$ is *never* an isomorphism.

(b) Let L be a field extension of K that is not finitely generated as a field extension. Prove that $L \otimes_K L$ is not a Noetherian ring.

Extra Credit 7. Let R be the polynomial ring in countably many indeterminates $y, x_1, \ldots, x_n, \ldots$ over a field K, let $I = (x_n : n \geq 1)R$, let $J = (y^n x_n : n \geq 1)R$, let $M = R/I$, and let $N = R/J$. Let $W = \{y^t : t \in \mathbb{N}\} \subseteq R$. Is $W^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$? Prove your answer.

Extra Credit 8. Let $R = K[x]/(x^n)$, where K is a field, x an indeterminate, and $n \geq 1$.

(a) Prove that $\mathrm{Hom}_K(R, K) \cong R$ as an R-module.

(b) Show that $\mathrm{Hom}_R(\_\,, R)$ and $\mathrm{Hom}_K(\_\,, K)$ are isomorphic functors from R-modules to R-modules. [By (a), the former is $\mathrm{Hom}_R\big(\_\,, \mathrm{Hom}_K(R,\, K)\big)$.]

(c) Prove that if M is an R-module and $R \to M$ is injective, then R is a direct summand of M as an R-module.

(d) Show that every epimorphism from R is surjective.