1. Let $\theta : F \oplus G \twoheadrightarrow A$ by $\theta(f \oplus g) = \alpha(f) + \beta(g)$. Call the kernel Q. We may restrict the projection map $F \oplus G \twoheadrightarrow F$ that maps $f \oplus g$ to f to Q. The map $\pi : Q \to F$ obtained is onto, because given $f \in F$ there exists $g \in G$ such that $\beta(g) = -\alpha(f)$, because β is surjective. The kernel of π consists of elements $0 \oplus g$ such that $\beta(g) = 0$, i.e., it is $0 \oplus N \cong N$. Thus, we have an exact sequence $0 \to N \to Q \to F \to 0$, and since F is projective this sequence splits. Thus, $Q \cong N \oplus F$. Similarly, $Q \cong M \oplus G$. $\square$

2. (a) The given bilinear map corresponds to a K-linear map $U \otimes_K V \to W$, which yields an L-linear map $\theta : L \otimes_K (U \otimes_K V) \to L \otimes_K W = W_L$. This map sends $c \otimes (u \otimes v)$ to $c \otimes B(u, v)$. By class results, $U_L \otimes_L V_L \cong (U \otimes_K L) \otimes_L (L \otimes_K V) \cong U \otimes_K \big(L \otimes_L (L \otimes_K V\big) \cong U \otimes_K \big((L \otimes_L L) \otimes_K V\big) \cong U \otimes_K (L \otimes_K V) \cong L \otimes_K (U \otimes_K V)$. With this identification, θ yields an L-linear map $U_L \otimes_L V_L \to W_L$. Moreover, $(c \otimes u)(d \otimes v) \mapsto (cd) \otimes (u \otimes v)$, and it follows that the corresponding bilinear map B_L sends $(c \otimes u, d \otimes d) \mapsto cd \otimes B(u, v)$. $\square$

(b) Choose bases $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ for U, V, W, respectively over K. These may be infinite. Then the $\{1 \otimes u : u \in \mathcal{U}\}$ give a basis for U_L over L, and similarly for V_L and W_L. Suppose that $B_L(u', v') = 0$ for nonzero $u' \in U_L$, $v' \in V_L$. Write u' as an L-linear combination of the specified basis elements, and similarly for v'. Suppose that $u' = \sum_{i=1}^r c_i \otimes u_i$, where $u_1, \dots, u_r$ are distinct elements of $\mathcal{U}$ and $v' = \sum_{j=1}^s d_j \otimes v_j$, where $v_1, \dots, v_s$ are distinct elements of $\mathcal{V}$. We may assume that all the c_i and d_j are nonzero. Choose sufficiently many elements $w_1, \dots, w_t$ of $\mathcal{W}$ that the values of all the $B(u_i, v_j)$ are in the K-span of $w_1, \dots, w_t$. We may replace u' by $(1/c_1)u'$ and v' by $(1/d_1)v'$ and so assume that $c_1 = 1$ and $d_1 = 1$. Then we have $B(u_i, v_j) = \sum_{k=1}^t a_{ijk} w_k$ for $1 \le i \le r$, $1 \le j \le s$, where the a_{ijk} are in K. Then $B_L(u', v') = 0$ means that $(\sum_{k=1}^t \sum_{1 \le i \le r, 1 \le j \le s} a_{ijk} c_i d_j) \otimes w_k = 0$, and so we have a system of equations $x_1 - 1 = 0, y_1 - 1 = 0$ and $\sum_{1 \le i \le r, 1 \le j \le s} a_{ijk} x_i y_j = 0$, $1 \le k \le t$, that has the solution $x_i = c_i$, $1 \le i \le r$, $y_j = d_j$, $1 \le j \le s$, over L. We claim that these equations, which have coefficients in K, must have a solution in K. Otherwise, by Hilbert's Nullstellensatz, the polynomials on the left hand sides generate the unit ideal in $K[x_1, \dots, x_r, y_1, \dots, y_s]$. But then they also generate the unit ideal in $L[x_1, \dots, x_r, y_1, \dots, y_s]$, and that would preclude a solution in L. Choose a solution in K, say $x_i = \gamma_i$, $1 \le i \le r$ with $\gamma_1 = 1$, and $y_j = \delta_j$, $1 \le j \le s$ with $\delta_1 = 1$. Let $u = \sum_{i=1}^r \gamma_i u_i$ and $v = \sum_{j=1} \delta_j v_j$. Then $u \ne 0$, $v \ne 0$, but the coefficient of every w_k in $B(u, v)$ is $\sum_{1 \le i \le r, 1 \le j \le s} a_{ijk} \gamma_i \delta_j$, which is 0 for every k. Hence, $B(u, v) = 0$, a contradiction. $\square$

3. (a) This follows by letting $U = V = W = D$ in **2.**, part (b) and $B(u, v) = uv$.

(b) Let L be the fraction field of C. Since K is a field, D is K-free, and so K-flat. Hence, $C \otimes_K D$ injects into $L \otimes_K D$, which is a domain by part (a).

4. (a) If $m = 0$, $R/m \cong R$ and the result is obvious. If $f \ne 0$ is a nonzero element of m, and R/m is flat, then $\alpha : (R/m) \otimes_R fR \to R/m \otimes_R R \cong R/m$ is injective. By Nakayama's lemma, $R/m \otimes fR \ne 0$ (or else $fR = 0$), and is generated by $1 \otimes f$. But $1 \otimes f$ maps to the image of f in R/m, which is 0, and so the injective map α is 0, a contradiction. Thus, we must have $m = 0$. $\square$

(b) If R is zero-dimensional and reduced, to prove that M is flat it suffices to prove that M_P is flat over R_P for all primes P. But every R_P is a zero-dimensional reduced quasilocal ring, i.e., a field, and over a field every module is free and, hence, flat. Conversely if every R-module is flat then for every prime P, R/P is flat, and so R_P/PR_P is R_P-flat. By part (a), every R_P must be a field. This implies that every prime ideal is minimal, and so R is zero-dimensional, and since every R_P is reduced, R is reduced. $\square$

5. The compositions will give surjections $M \to M$ and $N \to N$. Thus, it suffices to prove that a surjection from a finitely generated module M to itself is injective: it follows that each of the surjections is injective, and, hence, an isomorphism. We give two proofs: the second does not need that R be Noetherian (the result is true without this hypothesis).

First proof. Let $f : M \to M$ be surjective, and let N_k denote the kernel of f^k (k-fold iterated composition). Then $N_1 = f^{-1}(0)$ and $f_{k+1} = f^{-1}(N_k)$. If $N_1 \neq 0$, then N_2 is strictly larger than N_1 (since the map is surjective, the image of N_2 is N_1, while the image of N_1 is 0). By a straightforward induction, N_{k+1} is strictly larger than N_k for all k: it must map onto N_k, while N_k maps onto N_{k-1}. This contradicts ACC for M. $\square$

Second proof. Let $R[x]$ be the polynomial ring in one variable over R. We can extend the R-module structure to on M to $R[x]$ by letting $xm = f(m)$ for all $m \in M$. Therefore, there is no loss of generality if we assume the surjection arises from multiplication by an element f of the ring (replace R by $R[x]$ while keeping M the same). Let N be the kernel of multiplication by f. If it is nonzero, we can choose a prime P of R such that $N_P \neq 0$. Then multiplication by f is surjection of $M_P \to M_P$ but is not injective. If f is a unit of R_P, multiplication by f is injective. Therefore f must be in PR_P. But then the surjectivity means that $fM_P = M_P$, and so $(PR_P)M_P = M_P$. Since M is finitely generated, so is M_P, and this contradicts Nakayama's lemma. $\square$

6. (a) $K \hookrightarrow L \hookrightarrow R$ and we may assume without loss of generality that $K \subseteq L \subseteq R$. Similarly, we may assume that $K \subseteq L \subseteq S$. Let $u \in L - K$. Then $1, u$ extends to a basis for R over K and likewise $1, u$ extends to a basis for S over K. Hence, the elements $u \otimes 1$ and $1 \otimes u$ are distinct elements in a basis for $R \otimes_K S$. But then $u \otimes 1 - 1 \otimes u$ is a nonzero element in the kernel $R \otimes_K S \twoheadrightarrow R \otimes_L S$. $\square$

(b) Since L is infinitely generated over $K = K_0$, we may choose an infinite sequence of elements $a_1, \ldots, a_n, \ldots$ such that for all $n \geq 1$, $a_{n+1} \notin K_n = K(a_1, \ldots, a_n)$. That is, $K \subset K_1 \subset \cdots \subset K_n \subset \cdots$ is an infinite strictly ascending tower of subfields of L. Let J_n denote the ideal that is the kernel of surjection $L \otimes_K L \to L \otimes_{K_n} L$. Then J_{n+1}/J_n may be identified with the kernel of the surjection $L \otimes_{K_n} L \to L \otimes_{K_{n+1}} L$, which, by part (a), is nonzero. Hence, the J_n form a strictly ascending chain of ideals of $L \otimes_K L$, and so this ring is not Noetherian. $\square$

EC7. N has a K-basis consisting of those monomials such that the exponent on y is strictly smaller than the smallest j such that x_j occurs with a positive exponent in the monomial. The annihilator of x_j is spanned by all such monomials such that y^j or a higher power of y occurs. It follows that no element of N is killed by all of the x_j. This implies that $\mathrm{Hom}_R(M, N) = 0$, and therefore so is $W^{-1}\mathrm{Hom}_R(M, N)$. After localization at W, both M, and N become isomorphic to $R[1/y]/IR[1/y] \cong K[y, 1/y]$, and there are nontrivial maps, such as the isomorphism. Therefore, the two are different. $\square$

EC8. (a) R has K-basis $1, x, x^2, \ldots, x^{n-1}$, while $\mathrm{Hom}_R(K)$ has K-basis $\delta_0, \delta_1, \ldots, \delta_{n-1}$, where δ_i is the linear functional whose value on x^i is 1 and which is 0 on other the other x^j, $0 \leq j \leq n-1$. The R-module structure on the linear functionals is such that the value of sF on r is $F(sr)$. It follows that $\delta_i = x^{n-1-i}\delta_{n-1}$, so that δ_{n-1} generates $\mathrm{Hom}_K(R, K)$, which is therefore a cyclic module, and so has the form R/I. Since $\dim_K(R/I) = n$, the same as $\dim_K R$, we must have $I = 0$. Thus, $\mathrm{Hom}_R(R, K) \cong R$. $\square$

(b) By (a), $\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}(M, \mathrm{Hom}_K(R, K)), \cong \mathrm{Hom}_R(M \otimes_R R,\, K)$ (by the stronger form of the adjointness of $\otimes$ and Hom discussed in class), which is $\cong \mathrm{Hom}_R(M,\, K)$. $\square$

(c) Since $R \to M$ is injectve, $\mathrm{Hom}_K(M,\, K) \to \mathrm{Hom}_K(M,\, K)$ is surjective (every injection splits in the category of K-vector spaces), which implies that $\mathrm{Hom}_R(M,\, R) \to \mathrm{Hom}_R(R, R)$ is surjective, using part (b). The map $M \to R$ that restricts to id_R is a splitting. $\square$

(d) Suppose that $R \to S$ is an epimorphism. Let R' be the image of R in S. Then R' also has the form $K[x]/(x^h)$, $h \leq n$, and $R' \hookrightarrow S$ is an epimorphism. Therefore, it suffices to show that an injective epimorphism from a ring of the form $K[x]/(x^n)$ is an isomorphism. Note that by part (c), we have a splitting $S = R \oplus M$ as R-modules. By the criterion discussed in class, $R \to S$ is an epimorphism iff $S \otimes_R S \to S$ is an isomorphism. But $(R \oplus M) \otimes_R (R \oplus M) \cong R \oplus (R \otimes_R M) \oplus (M \otimes_R R) \oplus (M \otimes_R M)$, and, if $M \neq 0$, the map to $S = R \oplus M$ identifies both of the distinct nonzero summands $R \otimes_R M$ and $M \otimes_R R$ with M, and so is not injective, a contradiction. $\square$