Math 614, Fall 2020 **Problem Set #5**
Due: Friday, December 4

1. Let R be any commutative ring. Suppose that R has Krull dimension n. Let $S = R[x]$ be the polynomial ring in one variable over R. Show that $n+1 \leq \dim(S) \leq 2n+1$. (This is known to be the sharpest statement that can be made. Note that the Noetherian case has much more constrained behavior.)

2. Let P be a prime ideal of R, a Noetherian ring, and let $W = R - P$. Let $P^{(n)}$ be the n th symbolic power of P, i.e., the contraction of $P^n R_P$ to R. Let J be $\bigcup_{w \in W} \mathrm{Ann}_R w$. Prove that $J = \bigcap_{n=1}^{\infty} P^{(n)}$.

3. Let R be a Noetherian ring, and let M be a finitely generated R-module. Let I be an ideal of R. Let $N = \mathrm{Ann}_M I$. Prove that $\mathrm{Ass}(M) = \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$. (By a class theorem, one has $\subseteq$. The problem is to prove $\supseteq$, which is false in general but true here.)

4. Let R be a Noetherian graded ring over $\mathbb{N}^h$ or $\mathbb{Z}^h$, where $h \geq 1$, and let M be a graded module (for $\mathbb{N}^h$ or $\mathbb{Z}^h$)

(a) Show that all associated primes of M are homogeneous ideals (for the $\mathbb{N}^h$ or $\mathbb{Z}^h$ grading). [Suggestion: if P is the annihilator of a nonzero element $u \in M$, replace u by a nonzero multiple $v = v_1 + \cdots + v_d$, where the v_i are the nonzero homogeneous components of v, such that d is as small as possible. Then show that all of the v_i have annihilator P.]

(b) Let R be a polynomial ring $K[x_1, \ldots, x_n]$ over a field K, and give R the $\mathbb{N}^n$-grading in which the $(a_1, \ldots, a_n)$-forms are the elements of the one-dimensional K-vector space $Kx_1^{a_1} \cdots x_n^{a_n}$. Let I be a proper ideal of R generated by monomials. Prove that every associated prime of I is generated by a subset of the variables, and that I has a primary decomposition in which every ideal is generated by monomials.

5. In the polynomial ring $R = K[w, x, y, z]$ over a field K, find an irredundant primary decomposition of $I = (wxyz^2, x^2, y^3, xy^2z)R$. Which associated primes are minimal and which are embedded (i.e., not minimal)? Which primary components are unique?

6. Let $\mathbb{R}$ be the real numbers, let $S = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1) = \mathbb{R}[x, y]$, where X and Y are indeterminates. Let $T = \mathbb{C} \otimes_{\mathbb{R}} S \cong \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$.

(a) Prove that $T \cong \mathbb{C}[u, 1/u]$ where $u = x + yi$, and so is a PID. Prove that S is a Dedekind domain.

(b) Let $m = (x - 1, y)S$, which is a maximal ideal of S. Show that mT is principal and exhibit a generator, $f \in T$. Prove that m is not principal by showing that f has no unit multiple in S. Prove that m^2 is principal.

(c) Prove that $m \oplus m \cong S \oplus S$.

Extra Credit 9. Let R be a Noetherian ring and let M be a finitely generated R-module whose associated primes are $P_1, \ldots, P_n$. Show that M has a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$ such that each factor M_{i+1}/M_i, $0 \leq i \leq s$, is a torsion-free module over one of the rings R/P_j.

(b) Show that if, moreover, S is R-flat and Noetherian, then the associated primes of $S \otimes_R M$ over S consist of all the associated primes of the modules S/P_jS, $0 \leq j \leq n$.

Extra Credit 10. Let R be a ring whose localizations at maximal ideals are all Noetherian.

(a) Show that if every element of R is contained in only finitely many maximal ideals, then R is Noetherian.

(b) Give an example where R is not Noetherian.

(c) Let $R = K[x_1, \dots, x_n, \dots]$ be the polynomial ring in an infinite sequence of variables over a field K. Partition the variables into sets $S_1, \dots, S_n, \dots$ such that S_n contains n of the variables. Let P_n be the prime ideal generated by S_n. Let $W = R - \bigcup_{n=1}^{\infty} P_n$. Prove that $W^{-1}R$ is Noetherian but has infinite Krull dimension.