1. Given a strictly ascending chain of primes $P_0 \subset \cdots \subset P_n$ in R, one has the strictly ascending chain $P_0R[x] \subsetneq \cdots \subsetneq P_nR[x] \subsetneq P_nR[x] + xR[x]$ in $R[x]$. This proves that $\dim(R[x]) \geq \dim(R) + 1$. For the other inequality we must show that given a strictly ascending chain $Q_0 \subset \cdots \subset Q_{2n+1}$ in S, there is a chain of length n in R. Let P_i be the contraction of Q_i. Then $P_0 \subseteq \cdots \subseteq P_{2n+1}$. The sequence of $2n+2$ primes breaks up into segments of consecutive elements that are equal. If each segment contains at most two elements, then we get at least $n+1$ distinct contractions, and these form a strictly ascending chain of length n, as required. Therefore, it suffices to show that each segment contains at most two elements, i.e., that a *chain* of primes in $R[x]$ all of whose elements lie over P in R contains at most two primes. The primes lying over P correspond, by an order-preserving map, to the primes of $(R-P)^{-1}R[x]/PR[x] \cong (R-P)^{-1}(R/P)[x] \cong \kappa[x]$, where $\kappa = (R-P)^{-1}(R/P)$ is the isomorphic to the fraction field of R/P. Since this is a PID, every nonzero prime is maximal, and chains in $\kappa[x]$ have length at most 1.

2. We have that $J = \mathrm{Ker}\,(R \to R_P)$, and so J is in the contraction of every ideal, even the 0 ideal, of R_P. Thus, J is in the intersection. If r is in the intersecton, then $r/1 \in \bigcap_n (PR_P)^n$, which is 0, since R_P is local, by a class result. It follows that $r \in J$. $\square$

3. Clearly, it suffices to show that $\mathrm{Ass}\,(M/N) \subseteq \mathrm{Ass}\,(M)$ to establish the other inclusion, Let $I = (f_1, \ldots, f_n)R$. Then we have a map $M \to M^{\oplus n}$ such that $u \mapsto (f_1u, \ldots, f_nu) \in M^{\oplus N}$. The kernel of this map is evidently N. It follows that the map induces an injection $M/N \hookrightarrow M^{\oplus n}$. Hence, $\mathrm{Ass}\,(M/N) \subseteq \mathrm{Ass}\,(M^{\oplus n})$. Since the latter has a finite filtration in which every factor is M, we have $\mathrm{Ass}\,(M^{\oplus n}) \subseteq$Ass (M), as required. $\square$

4. (a) Following the suggestion and notation in the problem, we see that if r is homogeneous and kills one component of the element and not another, multiplying by r produces a nonzero element with at least one fewer component. Hence, when the number of components is minimum, all components have the same annihilator. Since the annihilator of a homogeneous element is clearly graded, all the components have the same graded annihilator P. We now claim that P is the annihilator of v. Suppose that $rv = 0$. Totally order $\mathbb{N}^h$ or $\mathbb{Z}^h$ so that $(a_1, \ldots, a_h) < (b_1, \ldots, b_h)$ if $a_i < b_i$ for the smallest i such that $a_i \neq b_i$. Let $r = r_1 + \cdots + r_k$ be the decomposition of r into components, where r_1 lies in the graded piece with the smallest index. Assume that v_1 is in the graded piece of v with the smallest index. Then r_1v_1 is in a graded piece of M with a smaller index than any other r_iv_j, and so cannot be canceled. Hence, $r_1v_1 = 0$. But then r_1 kills all the v_j, and so $r_1v = 0$. This implies that $(r_2 + \cdots + r_k)v = 0$. It follows by induction on k that all the r_j kill v, and so $\mathrm{Ann}_R v$ is graded, as required. $\square$

(b) A graded prime will be generated by monomials. Since the only irreducible monomials are variables, the prime must contain a variable that is a factor of each monomial in it, and so is generated by a subset of the variables.

Next note that if an ideal J is generated by monomials in a subset S of the variables and contains a power of each of these variables, then it is primary to the prime P generated by S. Any associated prime of J must contain P, and must be generated by a subset of the variables. But it is clear that the variables not in S are not zerodivisors on J. Hence, P is the only associated prime of J, which implies that J is P-primary.

Given a primary decomposition for a monomial ideal I, it suffices to replace each primary component for a given associated prime P by one contained in it which is generated by monomials. We follow the idea in Problem 2, slightly modified. Given such a primary ideal Q for, say, the prime $P = (x_1, \ldots, x_h)$ (we may assume this form after renumbering the variables), we note that since the remaining variables are not zerodivisors on Q, for all large N we have that Q contains the monomial ideal $Q' = (I + P^N)R_y \cap R$, where $y = x_{h+1} \cdots x_n$. Thus, it suffices to show that Q' is P-primary. Note that Q may be obtained by taking each monomial in $I + P^n$ and replacing it by the monomial obtained by omitting those factors involving $x_{h+1}, \ldots, x_n$. Therefore, Q' is generated by monomials ini $K[x_1, \ldots, x_h]$, and contains a power of every x_j for $j \leq h$, which implies that it is P-primary. $\square$

Alternate: a finite intersection of monomial ideals is generated by the least common multiples, which are monomials, of all selections of generators, one from each ideal. Suppose the given ideal is $(\mu_1, \ldots, \mu_k)$ where $\mu_i = x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}$ (some of the exponents may be 0). If J is generated by the $k-1$ monomials other than μ_i, then $I = J + (x_i^{a_{i,1}} \cdots x_n^{a_{i,n}}) = \bigcap_{j=1}^{n} \big(J + (x_j^{a_{i,j}})\big)$. Iterating, we obtain the followng. For each sequence of integers $\beta = b_1, \ldots, b_k$ where $1 \leq b_j \leq n$, let $Q_\beta = (x_{b_i}^{a_{i,b_i}} : 1 \leq i \leq k)$. Then Q_β is generated by a set of powers of variables, and is primary. One has $I = \bigcap_\beta Q_\beta$. One can combine ideals with same radical as usual, by intersectiing them, and omit any terms not needed to obtain an irredundant primary decomposition by monomial ideals. $\square$

5. The intersection of two monomial ideals is the monomial ideal generated by least common multiples of monomial generators for the two ideals. From this it follows that
$(wxyz^2, x^2, y^3, xy^2z) =$
$(w, x^2, y^3, xy^2z) \cap (x, x^2, y^3, xy^2z)) \cap (y, x^2, y^3, xy^2z) \cap (z^2, x^2, y^3, xy^2z) =$
$(w, x^2, y^3, xy^2z) \cap (x, y^3) \cap (x^2, y) \cap (z^2, x^2, y^3, xy^2z) =$
$(w, x^2, y^3, xy^2z) \cap (x^2, y^3, xy) \cap (z^2, x^2, y^3, xy^2z)$. Except for the first, these are primary by the discussion in the solution to Problem 4(b). The first may be written as
$(w, x^2, y^3, x) \cap (w, x^2, y^3, y^2) \cap ((w, x^2, y^3, z) =$
$(w, x, y^3) \cap (w, x^2, y^2) \cap (w, x^2, y^3, z)$.
The first two may be intersected to give the ideal (w, x^2, y^3, xy^2). Hence,
$I = (w, x^2, y^3, xy^2) \cap (w, x^2, y^3, z) \cap (x^2, y^3, xy) \cap (z^2, x^2, y^3, xy^2z)$.
The second term can be omitted, but no other. Therefore $I = (w, x^2, y^3, xy^2) \cap (x^2, y^3, xy) \cap$
(z^2, x^2, y^3, xy^2z)
is an irredundant primary decomposition. Hence, the associated primes are (x, y), which is minimal, and that primary component is unique, as well as (x, y, w) and (x, y, z), which are embedded, and their primary components are not unique.

6. If we make the linear change of coordinates $u = x + yi$, $v = x - yi$, $\mathbb{C}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{C}[u, v]/(uv - 1) \cong \mathbb{C}[u, 1/u]$, as claimed. Since this is a PID, it is a Dedekind domain, and S is a Dedekind domain by the preceding problem. In T, $x = (u + v)/2 = (u^2 - 2u + 1)/2u$ and $y = (u - v)/2i = -i(u - v)/2 = -i(u^2 - 1)/2u$. The ideal they generate is the same as the ideal generated by $(u - 1)^2$ and $u^2 - 1$ ($2u$ and $-i/2$ are units), and since the GCD

is $u-1$, this is the ideal generated by $u-1$. If this ideal is principal in S, the generator, viewed in $\mathbb{C}[u, 1/u]$, must be a unit times $u-1$ in $\mathbb{C}[u, 1/u]$. Since the units of $C[u, 1/u]$ are the elements cu^n, where $c \in \mathbb{C} - \{0\}$, the issue is whether there exist $c \in \mathbb{C} - \{0\}$ and $n \in \mathbb{Z}$ such that $(*) \quad cu^n(u-1) \in \mathbb{R}[x, y]$. There are $\mathbb{C}$-homomorphisms θ and θ' of $T \to \mathbb{C}$ such that $\theta(x) = \theta'(x) = 0$ and $\theta(y) = 1$ (resp., $\theta'(y) = -1$). Both map $\mathbb{R}[x, y]$ into $\mathbb{R}$. Note that $\theta(u) = i$ and $\theta'(u) = -i$. Applying these to (*), we find that $ci^n(i-1) \in \mathbb{R}$ and that $c(-i)^n(-i-1) \in \mathbb{R}$. Taking the ratio, we have that $(-1)^n(-i-1)/(1-i) \in \mathbb{R}$, which is false. Thus, m is not principal. $\square$
Since $T = S + Si \cong S \oplus S$, $m \oplus m \cong m \otimes_S T \cong mT \cong T$ (since mT is principal) $\cong S \oplus S$. Finally, $m^2 = ((x-1)^2, (x-1)y, y^2) \subseteq (x-1)$ since $y^2 = 1 - x^2 = -(x-1)(x+1)$, and $x - 1 = (-1/2)(x^2 - 2x + 1 + y^2)$ (since $x^2 + y^2 = 1$), so that $m^2 = (x-1)S$.

EC9. (a) Choose a maximal associated prime $Q = P_{j_1}$ of M and let $M_1 = \mathrm{Ann}_M Q$. Then M_1 is killed by Q and may be regarded as a module over R/Q. It cannot have torsion elements: these would have strictly larger annihilator than Q in R, and would have multiples with a prime annihilator Q' strictly larger than Q. We next claim $\mathrm{Ass}\,(M/M_1) \subseteq \mathrm{Ass}\,(M)$. Let $Q = (f_1, \ldots, f_h)$. There is a map $M \to M^{\oplus h}$ sending $m \mapsto (f_1 m, \ldots, f_h m)$. The kernel of this map is M_1, which yields an injection $M/M_1 \hookrightarrow M^{\oplus h}$. Hence, $\mathrm{Ass}\,(M/M_1) \subseteq \mathrm{Ass}\,(M^{\oplus h}) = \mathrm{Ass}\,(M)$. We can now continue recursively to construct $M_2/M_1 \subseteq M/M_1$, $M_3/M_2 \in M/M_2$, in the same way. The process must terminate (i.e., eventually M/M_s is 0), since M has ACC. This produces a finite filtration with the required property. $\square$

(b) If $P \in \mathrm{Ass}\,(M)$ we have $R/P \hookrightarrow M$, and we may tensor with S to obtain $(R/P) \otimes_R S \cong S/PS \hookrightarrow S \otimes_R M$ which shows that $\mathrm{Ass}\,(S/PS)$ (over S) is contained in $\mathrm{Ass}\,(S \otimes_R M)$ (over S). For the other direction, note that $S \otimes_R M$ has a filtration by the submodules $S \otimes M_i$ with factors $S \otimes (M_{i+1}/M_i)$, whence $\mathrm{Ass}\,(S \otimes_R M)$ is contained in the union over i of the $\mathrm{Ass}\,(S \otimes M_{i+1}/M_i)$. Next note that if T is torsion-free and finitely generated over $D = R/P$, then T embeds in D^h. To see this, take a maximal submodule N of T that is D-free, say $N \cong T^{\oplus a}$. Then T/N must be T-torsion, for otherwise, if $u \in T$ represents an element that is not torsion, $N \oplus Tu$ is larger D-free submodule. Thus, each generator of T is multiplied into N by a nonzero element of D: the product d of these multipliers is such that $dT \subseteq N \cong D^{\oplus a}$. But $T \cong dT$, and so we have $T \hookrightarrow D^{\oplus a}$. This yields $S \otimes_R T \hookrightarrow (S \otimes_R D)^{\oplus a}$, and so every associated prime of $S \otimes_R T$ is an associated prime of $S \otimes_R D \cong S \otimes_R (R/P) = S/PS$, for $P \in \mathrm{Ass}\,(M)$, as required. $\square$

EC10. (a). Let I be an ideal of R. If I is not 0, choose f nonzero in R. It will suffice to show that I/fR in R/fR is finitely generated: f together with liftings of the generators will generate I. Thus, we can reduce to the case where R has only finitely many maximal ideals. For each maximal ideal m_i, choose a finite set of generators of the form $g_i j/1$ for IR_{m_i}. Let J be the ideal generated all the g_{ij}. Then $(I/J)R_m = 0$, for all of the maximal ideals of R, and so $I = J$. $\square$

(b) Let R be the ring of locally constant functions to a field on an infinite, compact, totally disconnected, Hausdorff space X. E.g., we may consider the locally constant real-valued functions on the set $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\} \cup \{0\}$. The prime ideals of this ring correspond to functions vanishing at one of these points: all primes are maximal and minimal. Every localization is a field, but this ring is not Noetherian since it has infinitely many minimal primes. $\square$

(c) We first show that every prime ideal of $S = W^{-1}R$ is contained in one of the P_nS. Suppose Q were a prime of R not contained in the union of the P_n that expands to a (proper) prime ideal of S. Let $f \neq 0$ be an element of Q. The variables contained in f are in only finitely, say at most $S_1, \dots, S_h$. For every $i \leq h$, choose $f_i \in Q$ that not in P_i. Among the variables choose distinct $y_1, \dots, y_h$ not in any occurring in g, in any of the f_i, and not in any of the sets $S_1, \dots, S_h$. Then $g = f + y_1 f_1 + \cdots y_h f_h$ has the property that when one expands, no terms cancel. There is a term from $y_i f_i$ not involving any variable from S_i, $1 \leq i \leq h$, and the terms of f are not in any P_n for any $n \geq h+1$. It follows that $g \in W$, a contradiction, for then $QS = S$. Hence, the maximal ideals of $W^{-1}R = S$ are simply the ideals P_nS. Each nonzero element of S is in only finitely many maximal ideals: if write the element as r/w, where $r \neq 0$ and $w \in W$, any maximal ideal of S that contains r/w contains r, and r is, at worst, in those P_nS such that P_n contains a variable that occurs in r. By part (a), to show that S is Noetherian is suffices to show that the localization of S at every P_nS is Noetherian, and this is the same as R_{P_n}. Let $x_1, \dots, x_n$ be the variables in P_n and call the other variables $y_1, y_2, y_3, \dots$. Let L be the field $K(y_i : i \geq 1)$. The elements occurring in denominators in L are inverted in R_{P_n}. Thus, if $B_n = L[x_1, \dots, x_n]$, $R_{P_n} \cong (B_n)_m$, where $m = (x_1, \dots, x_n)B$, and this is Noetherian since B