Math 615: Lecture of January 19, 2007

We give one example of how one starts to calculate a Gröbner basis in a specific instance. Let $g_1 = x_1^2x_2x_4 + x_3^4$ and $g_2 = x_1x_3x_4^2 + x_4^4$ be generators for an ideal I in $R = K[x_1,\, x_2,\, x_3,\, x_4]$ and suppose that we are using hlex as the monomial order. Then $\nu_1 = \mathrm{in}(g_1) = x_1^2x_2x_4$ and $\nu_2 = \mathrm{in}(g_2) = x_1x_3x_4^2$ are elements of $\mathrm{in}(I)$. To test whethre this is a Gröbner basis we calculate G_{12} and h_{12}. Here, $\Delta_{12} = \mathrm{GCD}(\nu_1,\, \nu_2) = x_1x_4$, and so

$$G_{12} = \frac{x_1x_3x_4^2}{x_1x_4}g_1 - \frac{x_1^2x_2x_4}{x_1x_4}g_2 = x_3x_4(x_1^2x_2x_4 + x_3^4) - x_1x_2(x_1x_3x_4^2 + x_4^4).$$

Note that the multiples of the two initial terms cancel. This simplifies to

$$G_{12} = x_3^5x_4 - x_1x_2x_4^4,$$

and no term is a multiple of ν_1 or ν_2, so that we may take $G_{12} = h_{12}$. We see that $x_1x_2x_4^4 \in \mathrm{in}(I)$, and we now consider whether $g_1,\, g_2, h_{12}$ might be a Gröbner basis.

We have yet to prove that the Buchberger criterion stated last time gives a sufficient condition for $g_1,\, \ldots\,, g_r$ to be a Gröbner basis. In fact, we shall prove a sharper result. Before stating the new version, we want to observe:

Lemma. *Let $g_1,\, \ldots\,, g_r$ be nonzero elements of F, with our usual notation conventions. If g_i and g_j are such that* all *of their terms involve the same element e_t of the ordered basis for F (this condition is automatically satisfied if $F = R$), and if the initial terms ν_i of g_i and ν_j of g_j are relatively prime (i.e., their GCD is e_t), then there is a standard expression for G_{ij} under division with respect to $g_1,\, \ldots\,, g_r$ such that the remainder $h_{ij} = 0$.*

The proof is left as an exercise: see Problem Set #1, problem 4.

We now state our sharpened version of the Buchberger criterion. $R = K[x_1,\, \ldots\,, x_n]$ is a polynomial ring over a field K, and $g_1,\, \ldots\,, g_r$ are nonzero generators of a module $M \subseteq F$, where F is a finitely generated free R-module with ordered basis. Let $\nu_j = \mathrm{in}(g_j)$ for $1 \le j \le r$. Consider any set of pairs of indices $i_\lambda < j_\lambda$ such that

(1) For every λ, ν_{i_λ} and ν_{j_λ} involve the same basis element of F.

(2) The standard relations $\theta_{i_\lambda j_\lambda}$ generate the module of relations on the terms $\nu_1,\, \ldots\,, \nu_r$.

For every λ, let

$$G_{i_\lambda j_\lambda} = \frac{\nu_j}{\mathrm{GCD}(\nu_{i_\lambda}, \nu_{j_\lambda})}g_i - \frac{\nu_i}{\mathrm{GCD}(\nu_{i_\lambda}, \nu_{j_\lambda})}g_j.$$

For every λ, let $h_{i_\lambda j_\lambda}$ be the remainder in *any* standard expression for $G_{i_\lambda j_\lambda}$ divided by $g_1,\, \ldots\,, g_r$. (One does not have to use the remainder that arises from the deterministic division algorithm.)

Theorem (sharpened Buchberger criterion). *Let notation be as in the preceding paragraph. A necessary and sufficient condition for $g_1, \ldots, g_r$ to be a Gröbner basis for M is that every $h_{i_\lambda j_\lambda} = 0$. If $F = R$, the condition is still sufficient if one only checks those λ such that $\mathrm{in}(g_{i_\lambda})$ and $\mathrm{in}(g_{j_\lambda})$ are not relatively prime. (More generally, one can omit the check for λ whenever g_{i_λ} and g_{j_λ} have all terms involving the same element of the ordered basis for F, and $\mathrm{in}(g_{i_\lambda})$ and $\mathrm{in}(g_{j_\lambda})$ are relatively prime.)*

The original statement used all pairs ν_i, ν_j involving the same element of the ordered basis in defining the h_{ij}. We have cut down the number of pairs needed in two ways. First, we only need to use enough pairs to get a basis for the relations on $\nu_1, \ldots, \nu_r$. It is often the case that one can use far fewer pairs. Second, when $F = R$, one can omit checking whether the remainder is 0 for any pair such that the monomials in the initial terms are relatively prime.

It is obvious that the condition in the sharpened Buchberger criterion is necessary. Before giving the proof of sufficiency, we make the following observation. Given a monomial order on F, for every element e_i in the ordered basis we get a monomial order on the ring, which we denote $>_t$, defined by the rule $\mu_1 > \mu_2$ precisely when $\mu_1 e_t > \mu_2 e_t$. In many cases all of these monomial orders on R are the same, but this need not be true in general. However, if $f \in R - \{0\}$, $g \in F - \{0\}$, and $\mathrm{in}(g)$ involves e_t, then

$$(\dagger) \quad \mathrm{in}(fg) = \mathrm{in}_{>_t}(f)\mathrm{in}(g).$$

To see why this is true, consider what happens when we calculate fg by applying the distributive law and taking all products of a term of f and a term of g. First consider only those terms that involve e_t. The specified term occurs, and it is clear that all other terms occurring that involve e_t are strictly smaller, so that it cannot be cancelled. Thus, it suffices to show that any product of a term μ_1 of f and a term $\mu_2 e_j$ of g with $j \neq t$ is also $\leq \mathrm{in}_{>_t}(f)\mathrm{in}(g)$ — the inequality must then be strict, because $j \neq t$. But $\mu_2 e_j \leq \mathrm{in}(g)$, and so

$$\mu_1 \mu_2 e_j \leq \mu_2 \mathrm{in}(g).$$

Since $\mu_2 \leq_t \mathrm{in}_{>_t}(f)$ by definition of $\mathrm{in}_{>_t}(f)$, we have that

$$\mu_2 \mathrm{in}(g) \leq \mathrm{in}_{>_t}(f)\mathrm{in}(g),$$

as required. $\square$

We are now ready to give the argument for sufficiency.

Proof of sufficiency for the sharpened Buchberger criterion. First, in the case where $F = R$, note that the omission of checking the remainder when the initial terms are relatively prime is justified by the Lemma above: one can always choose a standard expression for which the remainder is 0, and so checking those pairs is unnecessary.

Now suppose that all the $h_{i_\lambda j_\lambda} = 0$. We must prove that $g_1, \ldots, g_r$ is a Gröbner basis. We assume the contrary, and obtain a contradiction.

If the $g_1, \ldots, g_r$ are not a Gröbner basis, we can choose

$$f = \sum_{j=1}^{r} f_j g_j$$

such that $\mathrm{in}(f)$ is not a multiple of any of the ν_j. We fix one such element f for the remainder of the proof. Let ϕ denote the r-tuple $(f_1, \ldots, f_r)$. Consider those terms on the right such that $f_j \neq 0$ and for these the ones such that the monomial ν_ϕ corresponding to $\mathrm{in}(f_j g_j)$ is largest as j varies: there may be several values of j that give rise to the same largest monomial ν_ϕ. There are typically many ways to write f as a linear combination of $g_1, \ldots, g_r$. Choose such a representation in such a way that ν_ϕ is minimum. This is possible because the monomial ordering on F is a well-ordering. We simply write $\nu = \nu_\phi$.

We shall obtain a contradiction by proving that if $\mathrm{in}(f)$ is not a multiple of any ν_j, then we can find a different representation for f as a linear combination of the g_j such that the value of ν_ϕ is strictly smaller.

After renumbering the g_j, we may assume that a nonzero scalar multiple of ν is the initial term of $f_i g_i$ for $1 \leq i \leq k$, and not for $f_j g_j$ with $j > k$. Each of $f_{k+1} g_{k+1}, \ldots, f_r g_r$ only involves terms that are strictly smaller than ν. To complete the argument, it will suffice to show that $f_1 g_1 + \cdots + f_k g_k$ can be rewritten as a linear combination of $g_1, \ldots, g_r$ so that the initial term of every product occurring in the sum is $< \nu$.

Suppose that ν involves e_t. Observe that by the discussion on p. 2 leading to the displayed formula (†), we know that for $1 \leq i \leq k$, $\mathrm{in}(f_i g_i) = \mu_i \nu_i$, where $\mu_i = \mathrm{in}_{>_t}(f_i)$. Here, each $\mu_i \nu_i$ is a scalar multiple of ν. We consider two cases.

First case. Here, we assume that $\sum_{i=1}^{k} \mu_i \nu_i \neq 0$. In this case, the value of the sum is a nonzero scalar multiple of ν, and so the initial term of f is evidently a nonzero scalar multiple of ν as well. This is an immediate contradiction, because, up to multiplication by a nonzero scalar, ν is the same as $\mu_i \nu_i$ for $1 \leq i \leq k$, and so is a multiple of ν_i for $1 \leq i \leq k$. But this contradicts the assumption that ν is not a multiple of any ν_j.

Second case. We assume that $\sum_{i=1}^{k} \mu_i \nu_i = 0$. We may write $f_i = \mu_i + \widetilde{f_i}$ for $1 \leq i \leq k$, and then we have

$$f = \sum_{i=1}^{k} \mu_i g_i + \sum_{i=1}^{k} \widetilde{f}_i g_i + \sum_{j=k+1}^{r} f_j g_j.$$

All terms occurring in the second and third sums are $< \nu$. Therefore, it will suffice to show that the first term, $\sum_{i=1}^{k} \mu_i g_i$, can be rewritten as a linear combination $\sum_{j=1}^{r} q_j g_j$ in such a way that every $\mathrm{in}(q_j g_j) < \nu$: after combining terms, we will have a new representation for f with a smaller ν_ϕ.

Since

$$\sum_{i=1}^{k} \mu_i \nu_i = 0,$$

we have that

$$\sum_{i=1}^{k} \mu_i e_i$$

is a relation on $\nu_1, \dots, \nu_r$. This relation has the same degree as ν, in the sense that each of the products has the same degree in $\mathbb{N}^n$ as ν. It follows that it can be written as a linear combination of the $\theta_{i_\lambda j_\lambda}$. Moreover, we may think of $\theta_{i_\lambda j_\lambda}$ as having the same degree as $LCM(\nu_{i_\lambda}, \nu_{j_\lambda})$. We therefore have an equation

$$(\#) \quad \sum_{i=1}^{k} \mu_i e_i = \sum_{\lambda} \zeta_\lambda \theta_{i_\lambda j_\lambda}$$

where the sum is extended over the indices λ that are needed (we do not include summands with coefficient 0), and each ζ_λ is a term such that

$$\deg(\zeta_\lambda) + \deg(\theta_{i_\lambda j_\lambda}) = \deg(\nu).$$

We now apply to $(\#)$ the map $R^r \to R$ sending $e_1, \dots, e_r$ to $g_1, \dots, g_r$ respectively. This yields

$$(*) \quad \sum_{i=1}^{k} \mu_i g_i = \sum_{\lambda} \zeta_\lambda G_{i_\lambda j_\lambda}.$$

Here,

$$G_{i_\lambda j_\lambda} = \frac{\nu_{j_\lambda}}{\mathrm{GCD}(\nu_{i_\lambda}, \nu_{j_\lambda})} g_i - \frac{\nu_{i_\lambda}}{\mathrm{GCD}(\nu_{i_\lambda}, \nu_{j_\lambda})} g_j.$$

Here, the initial term of each summand on the right is the same

$$\nu_{j_\lambda} \frac{\nu_{i_\lambda}}{\mathrm{GCD}(\nu_{i_\lambda}, \nu_{j_\lambda}))} = \nu_{i_\lambda} \frac{\nu_{j_\lambda}}{\mathrm{GCD}(\nu_{i_\lambda}, \nu_{j_\lambda}))}$$

which is the same up to a nonzero scalar multiple as $\mathrm{LCM}(\nu_{i_\lambda}, \nu_{j_\lambda})$. Since the initial terms cancel, we have that

$$\mathrm{in}(G_{i_\lambda j_\lambda}) < \mathrm{LCM}(\nu_{i_\lambda}, \nu_{j_\lambda}),$$

and it follows that when we multiply by ζ_λ we have that

$$\mathrm{in}(\zeta_\lambda G_{i_\lambda j_\lambda}) < \nu.$$

By hypothesis, every $G_{i_\lambda j_\lambda}$ has a standard expression of the form $\sum_{j=1}^{r} q_j^\lambda g_j$ in which the initial term of each product in the sum is $\leq \mathrm{in}(G_{i_\lambda j_\lambda})$. We now substitute into $(*)$ above to obtain

$$\sum_{i=1}^{k} \mu_i g_i = \sum_{\lambda} \sum_{j=1}^{r} \zeta_\lambda q_j^\lambda g_j$$

and for all λ and j we have

$$\mathrm{in}(\zeta_\lambda q_j^\lambda g_j) \le \zeta_\lambda \mathrm{in}(G_{i_\lambda j_\lambda}) = \mathrm{in}(\zeta_\lambda G_{i_\lambda j_\lambda}) < \nu,$$

exactly as required. $\square$

Review of complexes and homology

By a *complex* over a ring A we mean a sequence of A-modules and A-module maps

$$(*) \quad \cdots \longrightarrow G_{t+1} \xrightarrow{d_{t+1}} G_t \xrightarrow{d_t} G_{t-1} \longrightarrow \cdots$$

indexed by $\mathbb{Z}$ such that for all t, $d_t \circ d_{t+1} = 0$. However, we shall frequently consider complexes such that $G_t = 0$ for all $t < 0$, and when we talk about the complex

$$\cdots \longrightarrow G_{t+1} \longrightarrow G_t \longrightarrow G_{t-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0$$

we mean to imply that all negative terms vanish. We refer to a complex in which $G_i = 0$ for $i < 0$ as a *left* complex. Likewise, when we talk about the complex

$$0 \longrightarrow G_k \longrightarrow G_{k-1} \longrightarrow \cdots \longrightarrow G_t \longrightarrow \cdots$$

we mean to imply that $G_i = 0$ for $i > k$. We frequently write $(G_\bullet, d_\bullet)$ or simpy $G_\bullet$ to described a complex as in $(*)$. Note that the condition that $d_t \circ d_{t+1} = 0$ is equivalent to the conditon that $\mathrm{Im}\,(d_{t+1}) \subseteq \mathrm{Ker}\,(d_t)$. We define the t th *homology* module of the complex $G_\bullet$, denoted $H_t(G_\bullet)$, by

$$H_t(G_\bullet) = \mathrm{Ker}\,(d_t)/\mathrm{Im}\,(d_{t+1}).$$

The module $\mathrm{Ker}\,(d_t)$ is referred to as the module of *cycles* in G_t (and its elements are called *cycles*, and the module $\mathrm{Im}\,(d_{t+1})$ is referred to as the module of *boundaries* in G_t (and its elements are called *boundaries*). In a complex, every boundary is a cycle.

The complex $G_\bullet$ is called *exact* at G_t or *exact* at the t th spot if, equivalently, $\mathrm{Im}\,(d_{t+1}) = \mathrm{Ker}\,(d_t)$ or $H_t(G_\bullet) = 0$. Thus, when we have exactness at G_t, every cycle in G_t is a boundary (the converse statement always holds in a complex). A complex is called *exact* if it is exact at every spot. Equivalently, a complex is exact if all of its homology modules vanish. A left complex $G_\bullet$ is called *acyclic* if $H_t(G_\bullet) = 0$ for all $t \ge 1$. This leaves the possibility that $H_0(G_\bullet) \ne 0$. In this $H_0(G) = G_0/\mathrm{Im}\,(G_1)$, and $H_0(G_\bullet)$ is sometimes referred to as the *augmentation* module for $G_\bullet$. The augmented complex

$$\cdots \to G_t \to \cdots \to G_1 \to G_0 \to H_0(G_\bullet) \to 0$$

is exact.

By a map $\phi = \phi_\bullet$ of complexes of A-modules $F_\bullet \to G_\bullet$ we mean a family of A-module maps $\phi_t : F_t \to G_t$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{t+1} & \longrightarrow & F_t & \longrightarrow & F_{t-1} & \longrightarrow & \cdots \\
\cdots & & \Big\downarrow{\scriptstyle \phi_{t+1}} & & \Big\downarrow{\scriptstyle \phi_t} & & \Big\downarrow{\scriptstyle \phi_{t-1}} & & \cdots \\
\cdots & \longrightarrow & G_{t+1} & \longrightarrow & G_t & \longrightarrow & G_{t-1} & \longrightarrow & \cdots
\end{array}$$

commutes. In this case, for each t there is a map of homology $H_t(F_\bullet) \to H_t(G_\bullet)$: if $z \in F_t$ is a cycle representing an element $[z] \in H_t(F_\bullet)$, the value of the induced map on $[z]$ is $[\phi_t(z)]$, which turns out to depend only on $[z]$. If all the ϕ_t are injective, $F_\bullet$ is called a *subcomplex* of $G_\bullet$, and if all the ϕ_t are surjective, $G_\bullet$ is called a *quotient complex* of $F_\bullet$.

One says that

$$0 \to E_\bullet \to F_\bullet \to G_\bullet \to 0$$

is a *short exact sequence of complexes* if for every t the sequence

$$0 \to E_t \to F_t \to G_t \to 0$$

is exact and, in this case, the *Snake Lemma* or *Serpent Lemma* asserts there is a long exact sequence of homology

$$\cdots \to H_{t+1}(G_\bullet) \to H_t(E_\bullet) \to H_t(F_\bullet) \to H_t(G_\bullet) \to H_{t-1}(E_\bullet) \to \cdots .$$

The maps $\partial_t : H_t(G_\bullet) \to H_{t-1}(E_\bullet)$ are referred to as the *connecting homomorphisms*. If $z \in G_t$ is a cycle representing a homology class $[z]$, we can choose an element $\widetilde{z} \in F_t$ that maps to it. The image y of $\widetilde{z}$ in F_{t-1} maps to 0 in G_{t-1}, and so there is an element $\widetilde{y} \in E_{t-1}$ that maps to y. It is easy to see that $\widetilde{y}$ is a cycle in E_{t-1}, and one defines $\partial_t([z]) = [\widetilde{y}]$. The definition turns out to be independent of the choices made.

Whenever $\phi_\bullet : E_\bullet \to F_\bullet$ is a subcomplex, we may form a quotient complex $G_\bullet$ by letting $G_t = \mathrm{Coker}\,(\phi_t) \cong F_t/\mathrm{Im}\,(E_t)$. The differential is induced by the differential on $F_\bullet$. Similarly, whenever $F_\bullet \to G_\bullet$ is a quotient complex, we may let $E_t = \mathrm{Ker}\,(\phi_t) \subseteq F_t$, and $E_\bullet$ is a subcomplex under the restriction of the differential on $F_\bullet$. In both these cases, the sequence $0 \to E_\bullet \to F_\bullet \to G_\bullet \to 0$ is a short exact sequence of complexes.

Some acyclic complexes and Diana Taylor's resolution for monomial ideals

Let B be any commutative ring. Let $k \in \mathbb{N}$ be fixed, and let G_t denote the free module B-module with free basis $u_{i_1,\ldots,i_{t+1}}$ where $1 \le i_1 < i_1 < \cdots < i_{t+1} \le k$, so that the generators of G_t are in bijective correspondence with the $t+1$ element subsets of $\{1, 2, \ldots, k\}$. In fact, if $\sigma = \{i_1, \ldots, i_{t+1}\}$ with $1 \le i_1 < \cdots < i_{t+1} \le k$, we shall also write u_σ for $u_{i_1,\ldots,i_{t+1}}$.

If $t > k-1$ or $t < 0$ we define $G_t = 0$. Then one forms a complex

$$0 \to G_{k-1} \to \cdots \to G_0 \to 0$$

by defining the differential on G_t as follows. Since G_t is free, it suffices to specify the differential d_t on a typical generator, and if σ is the set $\{i_1, \dots, i_{t+1}\}$ with

$$1 \le i_1 < \cdots < i_{t+1} \le k$$

then $d_t(u_\sigma) = \sum_{j=1}^{t+1}(-1)^j u_{\sigma - \{i_j\}}$. It is easy to check that $d_{t-1} \circ d_t = 0$. The point is that after applying both maps, one gets a sum of terms $\pm u_{\sigma - \{i_j, i_{j'}\}}$ as $j \ne j'$ run through all pairs of distinct integers in the set $\{1, \dots, t+1\}$. Each term occurs exactly twice, once when i_j is deleted first and then $i_{j'}$, and a second time when $i_{j'}$ is deleted first and then i_j. It easy to verify that the signs one gets on these two occurrences are opposite, so that all terms cancel.

For those familiar with simplicial homology, we remark that this complex is precisely the complex used to calculate the simplicial homology of a $(k-1)$-simplex. It is therefore well-known that:

Proposition. *For all $k \ge 1$, the complex $G_\bullet$ described above is acyclic and $H_0(G_\bullet) \cong B$. Moreover, if we augment $G_\bullet$ by letting $G_{-1} = Bu_\emptyset$, where the new differential maps every u_i to $u_\emptyset$, the complex*

$$0 \to G_{k-1} \to \cdots \to G_0 \to G_{-1} \to 0$$

is exact.

Proof. We shall give two elementary proofs of this. We leave it to the reader to check that the first statement implies the second.

In the first proof proceed by induction on k. If $k = 1$, the complex is simply

$$0 \to Bu_1 \to 0$$

and the result is clear. Suppse $k > 1$. In the general case, note that the complex $F_\bullet$ corresponding to the set $1, 2, \dots k-1$ is a subcomplex. The quotient complex has free generators indexed by subsets of $\{1, 2, \dots, k\}$ such that k is an element of the subset. These are in bijective correspondence with the subsets of $\{1, \dots, k-1\}$ (including the empty set), and this gives a complex isomorphic with the augmented complex of F except that degrees are shifted by 1. Thus, the quotient $G_\bullet/F_\bullet$ is not merely acyclic, but exact, because it is augmented, and the result is now immediate from the Snake Lemma. $\square$

We can also prove acyclicty as follows. Let $h_t : G_t \to G_{t+1}$ be the map that sends $u_\sigma \mapsto 0$ if $1 \in \sigma$ and to $U_{\{1\} \cup \sigma}$ otherwise. Then for every σ,

$$d_{t+1}\big(h_t(u_\sigma)\big) + h_{t-1}\big(d_t(u_\sigma)\big) = u_\sigma$$

for $t \ge 1$ (consider the cases where $1 \in \sigma$ and $1 \notin \sigma$ separately). Thus, $d_{t+1}h_t + h_{t-1}d_t$ is the identity map on G_t. Suppose that $z \in G_t$ is a cycle, where $t \ge 1$. Then

$$d_{t+1}\big(h(z)\big) + h_{t-1}\big(d_t(z)\big) = z.$$

Since $d_t(z) = 0$, $d_{t+1}\big(h(z)\big) = z$, so that every cycle z is a boundary for $t \geq 1$. It remains to check that $H_0(G_\bullet) = B$, which we leave as an informal exercise. $\square$

We next want to describe Diana Taylor's resolution of a monomial ideal. We emphasize that these resolutions are rarely minimal.

We can make use of an arbitrary base ring B. Let $A = B[x_1, \ldots, x_n]$ be a polynomial ring and let $\mu_1, \ldots, \mu_k$ be monomials in A. We shall describe the resolution as an $\mathbb{N}^n$-graded complex: the generators of the free modules will typically have degrees in $\mathbb{N}^n$. The free basis of the t th free module will consist of elements $U_{i_1,\ldots,i_{t+1}}$ indexed by sequences $1 \leq i_1 < \cdots < i_{t+1} \leq k$, just as before. We give this generator the same degree as $\mathrm{LCM}(\mu_{i_1}, \ldots, \mu_{i_{t+1}})$. Then F_t is spanned as a free B-module by the elements $\mu U_{i_1,\ldots,i_{t+1}}$, where μ is a monomial in A, and this element will have the same degree as $\mu\mathrm{LCM}(\mu_{i_1}, \ldots, \mu_{i_{t+1}})$. If $\sigma = \{i_1, \ldots, i_{t+1}\}$, it will be convenient to write U_σ for $U_{i_1,\ldots,i_{t+1}}$, and to define

$$\mathrm{LCM}(\mu_\sigma) = \mathrm{LCM}(\mu_{i_1}, \ldots, \mu_{i_{t+1}}).$$

We can now define the differential on $F_\bullet$ by the rule

$$d_t(U_\sigma) = \sum_{j=1}^{t+1} (-1)^j \frac{\mathrm{LCM}(\mu_\sigma)}{\mathrm{LCM}(\mu_{\sigma-\{i_j\}})} U_{\sigma-\{i_j\}}$$

Note that this formula preserves degrees. Let $I = (\mu_1, \ldots, \mu_k)A$, and augment the complex $F_\bullet$ by the map $F_0 \twoheadrightarrow I$ such that $U_i \mapsto \mu_i$ for $1 \leq i \leq k$. Note that the maps d_t preserve degree.

Theorem (Diana Taylor). *Let $A = B[x_1, \ldots, x_n]$, $\mu_1, \ldots, \mu_k$, I, and $F_\bullet$ be as above. Then*

$$0 \to F_{k-1} \to \cdots \to F_0 \to 0$$

is an acyclic complex that gives a free resolution of I, i.e., the augmented complex

$$0 \to F_{k-1} \to \cdots \to F_0 \to I \to 0$$

is exact.