Math 615: Lecture of January 22, 2007

Proof. Because the maps are degree preserving, it suffices to prove that the complex is exact in each degree α. In fact, the full complex

$$0 \to F_{k-1} \to \cdots \to F_0 \to I \to 0$$

is the direct sum of the homogeneous subcomplexes

$$(*_\alpha) \quad 0 \to [F_{k-1}]_\alpha \to \cdots [F_0]_\alpha \to [I]_\alpha \to 0.$$

It will therefore suffice to prove that each of the complexes $(*_\alpha)$ is exact.

Noe the following: the contribution to $[F_t]_\alpha$ from AU_σ is 0 unless $\mathrm{LCM}(\mu_\sigma)$ divides x^α. In this case, there is a unique monomial ν_σ such that $\nu_\sigma \mathrm{LCM}(\mu_\sigma) = x^\alpha$, so that $[F_t]_\alpha$ is the free B-module generated by the elements $\nu_\sigma U_\sigma$ such that $\mathrm{LCM}(\mu_\sigma)$ divides x^α. Let $\mu_{j_1}, \ldots, \mu_{j_h}$ with $j_1 < \cdots < j_h$ be the generators of I that divide α. Then $\mathrm{LCM}(\mu_\sigma)$ divides x^α iff μ_i divides x^α for every $i \in \sigma$ iff $\sigma \subseteq \{j_1, \ldots, j_h\}$.

Therefore, if $x^\alpha \notin I$, every $[F_t]_\alpha = 0$ and $[I]_\alpha = 0$, while if $x^\alpha \in I$, and $\mathcal{S}_\alpha = \{\mu_{j_1}, \ldots, \mu_{j_h}\}$ is the set of generators of I that divide x^α, $[F_t]_\alpha$ is the free B-module on the elements $\nu_\sigma U_\sigma$ such that $\sigma \subseteq \mathcal{S}_\alpha$ and σ is a set with $t+1$ elements. The set $\mathcal{S}_\alpha$ is in bijective correspondence with $\{1, \ldots, h\}$, with μ_{j_i} corresponding to i, and for each $t+1$ element subset τ of $\{1, \ldots, h\}$ we may let u_τ denote the element $\nu_\sigma U_\sigma \in [F_t]_\alpha$, where σ is the $t+1$ element subset of $\mathcal{S}_\alpha$ corresponding to τ. The complex $[F_t]_\alpha$ is then isomorphic to an augmented complex $G_\bullet$ over B of the form described at the bottom of p. 6 and on p. 7 of the Lecture Notes of January 19 (but with h replacing k), and so is exact by the Proposition on p. 7 of those notes. $\square$

Finding Hilbert-Poincaré series

Let M be a finitely generated module over $K[x_1, \ldots, x_n]$. When we consider the $\mathbb{N}^n$ grading on R, we shall allow $\mathbb{Z}^n$-gradings on M. When we consider the $\mathbb{N}$-grading on R, we shall allow $\mathbb{Z}$-gradings on M.

Note that, quite generally, when $H \subseteq H'$ is a subsemigroup of the additive semigroup H' and R is an H-graded ring, we can also view R as H'-graded by letting $R_{h'} = 0$ for $h' \in H' - H$. Therefore, we can consider H'-graded modules M over the H-graded ring R. In effect, the condition becomes that for $h \in H$ and $h' \in H'$, $R_h M_{h'} \subseteq M_{h+h'}$.

In our cases $H = \mathbb{N}^n$ and $H' = \mathbb{Z}^n$ or $H = \mathbb{N}$ and $H' = \mathbb{Z}$.

Because M is finitely generated over $R = K[x_1, \ldots, x_n]$, if $-B$ is the smallest integer such that some generator of M has a degree involving $-B$, then all nonzero homogeneous elements of M have degree $\geq -B$ in every coordinate: when we multiply by monomials in R, degrees can only increase.

When M is $\mathbb{Z}$-graded, this means that there only finitely many nonzero components of M in negative degree.

If $\alpha \in \mathbb{Z}^n$, we define $M(\alpha)$ (sometimes called M *twisted by* α or the α th *twist of* M) to be the $\mathbb{Z}^n$-graded module that is isomorphic to M as an R-module but with grading shifted so that for all $\beta \in \mathbb{Z}^n$,

$$[M(\alpha)]_\beta = M_{\alpha+\beta}.$$

One reason for introducing these shifted gradings is that in considering free resolutions of graded modules one often wants to use maps that preserve degree. In doing this, one may need to shift gradings even when working with free modules.

Consider one of the simplest possible examples, where $R = K[x]$ and $M = K[x]/xK[x]$, which has the free resolution:

$$0 \to K[x] \xrightarrow{x\cdot} K[x] \longrightarrow M \longrightarrow 0$$

The element $1 \in K[x]$ in the leftmost module maps to x in the copy of $K[x]$ to the right. If the map is to be degree-preserving, we need $1 \in K[x]$ to have degree 1. If the right hand copy of $K[x]$ has the usual grading, this means that the leftmost copy should be twisted by -1. The resolution is then

$$0 \to R(-1) \xrightarrow{x\cdot} R \to M \longrightarrow 0.$$

Note that $[R(-1)]_1 = [R]_{1+(-1)} = [R]_0 = K$, so that 1 has degree 1 in $R(-1)$. Typically, 1 has degree t in $R(-t)$ for all $t \in \mathbb{Z}$.

Also note that any finitely generated $\mathbb{Z}^n$-graded module M over $R = K[x_1, \ldots, x_n]$ has a twist $M(\alpha)$ with the property that $[M(\alpha)]_\beta$ is a nonzero component only if $\beta \in \mathbb{N}^n$. If no generator involves a degree smaller than $-B$ in any component, we may take $\alpha = (-B, \ldots, -B)$. If β has any strictly negative entry, $\alpha + \beta$ has entry $< -B$, and $[M(\alpha)]_\beta = 0$.

We next define the Hilbert-Poincaré series $\mathfrak{P}^\mu_M(z_1, \ldots, z_n)$ of an $\mathbb{N}^n$-graded module M over $K[x_1, \ldots, x_n]$ (here, the superscript μ indicates that we are using the $\mathbb{N}^n$-graded version) by the formula

$$\mathfrak{P}^\mu_M(z_1, \ldots, z_n) = \mathfrak{P}^\mu_M(z) = \sum_{\alpha \in \mathbb{Z}^n} \dim_K([M]_\alpha) z^\alpha,$$

which *a priori* is an element of

$$\mathbb{Z}[[z_1, \ldots, z_n]](1/z_1 \cdots z_n).$$

However, we shall soon prove that these series are actually rational functions of $z_1, \ldots, z_n$.

We first consider the case of R itself. Then

$$\mathfrak{P}^{\mu}_{M}(z) = \sum_{\alpha \in \mathbb{N}^n} z^{\alpha} = (1 + z_1 + z_1^2 + \cdots)(1 + z_2 + z_2^2 + \cdots)\cdots(1 + z_n + z_n^2 + \cdots)$$

$$= \prod_{i=1}^{n} \frac{1}{1 - z_i} = \frac{1}{\prod_{i=1}^{n}(1 - z_i)}.$$

Note that if we have a short exact sequence of $\mathbb{Z}^n$-graded finitely generated modules and degree-preserving maps, say $0 \to M_2 \to M_1 \to M_0 \to 0$, then we get a short exact sequence of vector spaces

$$0 \to [M_2]_{\alpha} \to [M_1]_{\alpha} \to [M_0]_{\alpha} \to 0$$

for every α. It follows that

$$\mathfrak{P}^{\mu}_{M_1}(z) = \mathfrak{P}^{\mu}_{M_0}(z) + \mathfrak{P}^{\mu}_{M_2}(z).$$

More generally, given a finite exact sequence

$$0 \to M_h \to \cdots \to M_0 \to 0$$

of finitely generated $\mathbb{N}^n$-graded modules and degree preserving maps, we have that

$$\sum_{i=0}^{h} (-1)^i \mathfrak{P}^{\mu}_{M_i}(z) = 0.$$

This follows simply because the exact sequence of length h can be broken up into short exact sequences. Diana Taylor's resolution for monomial ideals now yields the following.

Theorem. *Let I be a monomial ideal with generators $\mu_1 = x^{\alpha_1}, \ldots, \mu_k = x^{\alpha_k}$ in $R = K[x_1, \ldots, x_n]$. Then $\mathfrak{P}^{\mu}_{R/I}(z)$ is a rational function of $z_1, \ldots, z_n$ whose numerator has integer coefficients and whose denominator is at worst $\prod_{i=1}^{n}(1 - z_i)$. More precisely, let Σ_t denote the sum of the least common multiples of the monomials $z^{\alpha_1}, \ldots, z^{\alpha_k}$ taken t at a time, for $0 \le t \le k$, where $\Sigma_0 = 1$. Then*

$$\mathfrak{P}^{\mu}_{R/I}(z) = \frac{\Sigma_0 - \Sigma_1 + \Sigma_2 - \cdots + (-1)^k \Sigma_k}{\prod_{i=1}^{k}(1 - z_i)}.$$

Proof. We can modify Diana Taylor's resolution slightly by putting it together with the short exact sequence $0 \to I \to R \to R/I \to 0$ to give

$$0 \to F_{k-1} \to \cdots \to F_0 \to R \to R/I \to 0.$$

Consequently, we have

$$(*) \quad \mathfrak{P}^{\mu}_{R/I}(z) = \mathfrak{P}^{\mu}_{R}(z) - \sum_{i=0}^{k-1}(-1)\mathfrak{P}^{\mu}_{F_i}(z).$$

F_i is the direct sum of copies of R, one for each $i+1$ element subset σ of $\{1, \ldots, k\}$, with the generator of R in degree $\mathrm{LCM}(\mu_\sigma) = x^{\beta_\sigma}$. The Hilbert-Poincare series of this cyclic free module is $z^{\beta}_{\sigma}\mathfrak{P}^{\mu}_{R}(z)$. It follows that the Hilbert-Poincaré series

$$\mathfrak{P}^{\mu}_{F_i}(z) = \Sigma_{i+1}\mathfrak{P}^{\mu}_{R}(z).$$

The result now follows from substituting this in $(*)$ and noting that

$$\mathfrak{P}^{\mu}_{R}(z) = \frac{1}{\prod_{i=1}^{n}(1-z_i)}. \qquad \square$$

Corollary. *If $F = R^s$ is free, for every monomial submodule M of F, F/M and M have Hilbert-Poincaré series that are rational functions whose numerator is a polynomial with integer coefficients and whose denominator is at worst $\prod_{i=1}^{n}(1-z_i)$.*

Proof. The monomial submodule is a direct sum of monomial ideals, one in each Re_i. $\square$

We want to consider what happens when the generators of F may have degrees shifted by twisting. The key point is that for any finitely generate $\mathbb{N}^n$-graded module M and any α,

$$\mathfrak{P}^{\mu}_{M(\alpha)}(z) = \sum_{\beta\in\mathbb{Z}^n}\dim_K([M]_{\alpha+\beta})z^{\beta} = z^{-\alpha}\sum_{\beta\in\mathbb{Z}^n}\dim_K([M]_{\alpha+\beta})z^{\alpha+\beta} = z^{-\alpha}\mathfrak{P}^{\mu}_{M}(z),$$

since as β runs through all of $\mathbb{Z}^n$, so does $\alpha+\beta$.

We now want use our monomial results to prove theorems about Hilbert-Poincaré series in the $\mathbb{N}$-graded case. As in the $\mathbb{Z}^n$-graded case,

$$\mathfrak{P}_{M(h)}(z) = z^{-h}\mathfrak{P}_{M}(z).$$

Next note:

Proposition. *Let M be a finitely generated $\mathbb{Z}^n$ graded modules over $K[x_1, \ldots, x_n]$. Then $\mathfrak{P}_M(z) = \mathfrak{P}^{\mu}_{M}(z, z. \ldots, z)$ (i.e., z is substituted for every z_i).*

In particular, $\mathfrak{P}_R(z) = \dfrac{1}{(1-z)^n}$. Hence, if $M \subseteq F = R^s$ (which includes the case $I \subseteq R$) is monomial then both $\mathfrak{P}_M(z)$ and $\mathfrak{P}_{F/M}(z)$ are rational functions in which the *numerator is a polynomial in z with integer coefficients and the denominator is at worst $(1-z)^n$.*

In general, for any finitely geneated $\mathbb{Z}^n$-graded module M, $\mathfrak{P}_M(z)$ is a rational function of z whose numerator is a polynomial in z with integer coefficients and whose denominator is, at worst, $z^B(1-z)^n$ for some $B \geq 0$.

Proof. If $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}_n$, we write $|\alpha|$ for $a_1 + \cdots + a_n$. Then for every integer i,

$$[M]_i = \bigoplus_{|\alpha|=i} [M]_\alpha,$$

and so

$$\dim_K([M]_i) = \sum_{|\alpha|=i} \dim_K([M]_\alpha),$$

and the result follows at once from this observation. The remaining statements are immediate. $\square$

We can now obtain a result for arbitrary finitely generated modules in the graded case.

Theorem. *Let N be any finitely generated $\mathbb{Z}$-graded module over $R = K[x_1, \ldots, x_n]$. Suppose that $u_1, \ldots, u_s$ are finitely many homogeneous generators of respective degrees $d_1, \ldots, d_s$. Think of R^s as $\bigoplus_{j=1}^s R(-d_j)$, and map $R^s \twoheadrightarrow N$ so that $1 \in R(-d_j)$, which has degree d_j, maps to u_j. This map preserves degrees, and the kernel M is an $\mathbb{N}$-graded submodule of R^s.*

Refine the $\mathbb{Z}$-grading on R^s to a $\mathbb{Z}^n$-grading, and choose a monomial order. Then N and $F/\mathrm{in}(M)$ have the same Hilbert-Poincaré series! Hence, the Hilbert-Poincaré series of N is a rational function of z with numerator that is a polynomial in z with integer coefficients and denominator at worst $z^B(1-z)^n$ for some $B \in \mathbb{N}$.

Proof. By the Theorem near the bottom of p. 2 of the Lecture of January 12, the monomials of F not in $\mathrm{in}(M)$ are a basis for F/M, and they are clearly a basis of homogeneous elements. Hence, the monomials of a given degree d are a K-vector space basis for $[F/M]_d$, and also for $[F/\mathrm{in}(M)]_d$, and so $\dim_K([F/M]_d) = \dim_K([F/\mathrm{in}(M)]_d)$ for all d. The first conclusion follows at once, and the second then follows as well because we already know the result in the monomial case. $\square$