Math 615: Lecture of January 26, 2007

Review of the theory of Krull dimension

We recall that the (*Krull*) *dimension* of a ring R, which need not be Noetherian, is the supremum of lengths k of strictly increasing chains $P_0 \subset P_1 \subset \cdots \subset P_{k-1} \subset P_k$ of chains of prime ideals of R. The *height* of a prime ideal P is, equivalently, either the supremum of lengths of strictly descending chains of primes whose first element is P, or the dimension of the quasilocal ring R_P (a *quasilocal* ring is a ring with a unique maximal ideal).

We have:

Proposition. *If J is an ideal of R consisting of nilpotent elements, then* $\dim(R) = \dim(R/J)$. *Hence, if I and I' are two ideals of R with the same radical,* $\dim(R/I) = \dim(R/I')$.

Proof. There is an order preserving bijection between primes of R and primes of R/J: every prime ideal P of R contains J, and we may let P correspond to P/J. The second statement now follows because if $J = \mathrm{Rad}\,(I) = \mathrm{Rad}\,(I')$, then R/J is obtained from either R/I or R/I' killing an ideal (J/I or J'/I) all of whose elements are nilpotent. $\square$

Theorem. *If $R \subseteq S$ is an integral extension of rings, then* $\dim(R) = \dim(S)$.

Proof. Given any finite strictly ascending chain of primes in R there is a chain of the same length in S by the going up theorem. Hence, $\dim(R) \leq \dim(S)$. On the other hand, given a strictly ascending chain of primes of S, we obtain a strictly ascending chain of primes in R by intersecting its elements with R. The intersections with R of comparable but distinct primes of S are distinct by the lying over theorem. $\square$

If R is Noetherian, every prime has finite height. In fact:

Krull Height Theorem. *If R is Notherian and $I \subseteq R$ is generated by n elements, the height of any minimal prime P of R is at most n. Moreover, every prime ideal of height n is a minimal prime of an ideal generated by n elements.*

By a local ring (R, m, K) we mean a Noetherian ring with a unique maximal ideal m such that $K = R/m$.

Corollary. *If R is a local ring, the dimension of R (which is the same as the height of m) is the least number n of elements $x_1, \ldots, x_n \in m$ such that m is the radical of $(x_1, \ldots, x_n)R$.*

A set of n elements as described above is called a *system of parameters* for the local ring R. When R is zero-dimensional, the system of parameters is empty.

Corollary. *If $f \in m$, where (R, m, K) is local, then* $\dim(R/fR) \geq \dim(R) - 1$.

Proof. Choose a system of parameters for R/fR that are the images of elements $x_2, \ldots, x_s$ in m, where $s - 1 = \dim(R/xR)$. Since m/fR is nilpotent on $(x_2, \ldots, x_s)$, we have that m is nilpotent on $(f, x_2, \ldots, x_s)R$. Therefore, $\dim(R) \leq s = \dim(R/fR) + 1$. $\square$

Theorem. *Let R be a domain finitely generated over a field K. The dimension n of R is the transcendence degree of its fraction field over K. Every maximal ideal of R has height n, and for any two primes $P \subseteq Q$, a maximal ascending chain of primes from P to Q (also called a* saturated chain *from P to Q) has length equal to* $\text{height}(Q) - \text{height}(P)$.

When R is finitely generated over a field K, it is an integral extension of a polynomial subring, by the Noether normalization theorem. This suggests why the statements in this Theorem ought to be true, and a proof can be based on this idea.

Krull dimension for modules

If M is a finitely generated module over a Noetherian ring R, we define the (*Krull*) *dimension* of M to be the Krull dimension of R/I, where $I = \text{Ann}_R M$ is the annihilator of I. We make the convention that the Krull dimension of the 0 ring is -1, and this means that the Krull dimension of the 0 module is also -1. Recall that the *support* of M, denoted $\text{Supp}(M)$ is

$$\{P \in \text{Spec}(R) : M_P \neq 0\}.$$

Also recall:

Proposition. *If M is a finitely generated module over a Noetherian ring R,* $\text{Supp}(M) =$ *$V(I)$, the set of prime ideals containing $I = \text{Ann}_R M$.*

Proof. Let $u_1, \ldots, u_k$ generate M. Then the map $R \to M^k$ that sends $r \mapsto (ru_1, \ldots, ru_k)$ has kernel precisely I, which yields an injection $R/I \hookrightarrow M^k$. If $I \subseteq P$, then $(R/I)_P \neq 0$ injects into $(M^k)_P \cong (M_P)^k$, and so $M_P \neq 0$. Conversely, if $f \in I - P$, then M_P is localization of M_f, which is 0 since $fM = 0$. $\square$

Recall that a prime ideal is an *associated* prime of M if there is an injection $f : R/P \hookrightarrow M$. It is equivalent to assert that there is an element $u \in M$ such that $\text{Ann}_R u = P$. The set of associated primes of M is denoted $\text{Ass}(M)$. By a theorem, $\text{Ass}(M)$ is finite.

Proposition. *Let R be a Noetherian ring and let M be a finitely generated R-module.*

(a) *The dimension of M is* $\sup\{\dim(R/P) : P \in \text{Supp}(M)\}$.

(b) *The dimension of M is* $\sup\{\dim(R/P) : P$ *is a minimal prime of* $M\}$.

(c) *The dimension of M is* $\sup\{\dim(R/P) : P \in \text{Ass}(M)\}$.

(d) *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence. Then* $\dim(M) = \max\{\dim(M'), \dim(M'')\}$.

(e) *If* $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k = M$ *is a finite flitration of* M*, then* $\dim(M) = \sup\{\dim(M_{i+1}/M_i : 0 \leq i \leq k-1\}$.

Proof. (a) and (b). Since $\mathrm{Supp}(M)$ is $V(I)$, the assertion comes down to the statement that $\dim(R/I) = \sup\{\dim(R/P) : I \subseteq P\}$. This is clear, since I has only finitely many minimal primes $P_1, \ldots, P_h$, and so $\dim(R/I)$ is the supremum of the integers $\dim(R/P_j)$ where $1 \leq j \leq h$.

(c) The minimal primes of M (equivalently, of the support of M) are the same as the minimal primes P of I. As in the proof of the preceding Proposition we have $R/I \hookrightarrow M^k$, and then

$$P \in \mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(M^k) = \mathrm{Ass}(M),$$

so that every minimal prime of I is in $\mathrm{Ass}(M)$. On the other hand, if $P \in \mathrm{Ass}(M)$ then $R/P \hookrightarrow M$ and so I kills R/P, i.e., $I \subseteq P$. Part (c) follows at once.

(d) Let I', I, and I'' be the annihilators of M', M, and M'' respectively. Then $I \subseteq I'$ and $I \subseteq I''$, so that $I \subseteq I' \cap I''$. If $u \in M$, then $I''u \subseteq M'$ (since I' kills $M/M' = M''$), and so I' kills $I''u$, i.e., $I'I''u = 0$. This implies that $I'I'' \subseteq I$. Now $(I' \cap I'')^2$ is generated by products fg where $f, g \in I' \cap I''$. Think of f as in I' and g as in I''. It follows that $(I' \cap I'')^2 \subseteq I'I'' \subseteq I' \cap I''$, so that $\mathrm{Rad}(I' \cap I'') = \mathrm{Rad}(I'I'')$, and we have that $\mathrm{Rad}(I) = \mathrm{Rad}(I'I'')$ as well. The result now follows from part (a) and the fact that $V(I'I'') = V(I') \cup V(I'')$.

(e) We use induction on the length of the filtration. The case where $k = 1$ is obvious, and part (d) gives the case where $k = 2$. If $k > 2$, we have that $\dim(M) = \max\{\dim(M_{k-1}, M_k/M_{k-1})$ by part (d), and and

$$\dim(M_{k-1}) = \sup\{\dim(M_{i+1}/M_i : 0 \leq i \leq k-2\}$$

by the induction hypotheis. $\square$

Remark. Let $M \neq 0$ be a finitely generated module over an arbitrary ring R. Then M has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k$$

such that every factor M_{i+1}/M_i, where $0 \leq i \leq k-1$, is a cyclic module. In fact if $u_1, \ldots, u_k$ generate M, we may take $M_i = Ru_1 + \cdots + Ru_i$, $0 \leq i \leq k$. If R is Noetherian, we can find such a filtration such that every M_{i+1}/M_i is a prime cyclic module, i.e., has the form R/P_i for some prime ideal I of R. One first chooses u_1 such that $\mathrm{Ann}_R u_1 = P_1$ is prime in R. Let $M_1 = Ru_1 \subseteq M$. Proceeding recursively, suppose that $u_1, \ldots, u_i$ have been chosen in M such that, with $M_j = Ru_1 + \cdots + Ru_j$ for $1 \leq j \leq i$, we have that $M_j/M_{j-1} \cong R/P_j$ with P_j prime. If $M_i = M$ we are done. If not we can choose $u_{i+1} \in M$ such that the annihilator of its image in M/M_i is a prime ideal P_{i+1} of R. Then $M_{i+1}/M_i \cong R/P_{i+1}$: in particular, the inclusion $M_i \subset M_{i+1}$ is strict. The process must terminate, since M has ACC. This means that evenutally we reach M_k such that $M_k = M$. For this type of filtration, it follows from part (e) of the Proposition above that we have

$$\dim(M) = \sup\{\dim(R/P_i) : 1 \leq i \leq k\}.$$

The graded case

This section contains several results that are useful in studying dimension theory in the graded case.

Proposition. *Let M be an $\mathbb{N}$-graded or $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded or $\mathbb{Z}$-graded Noetherian ring S. Then every associated prime of M is homogeneous. Hence, every minimal prime of the support of M is homogeneous and, in particular the associated (hence, the minimal) primes of S are homogeneous.*

Proof. Any associated prime P of M is the annihilator of some element u of M, and then every nonzero multiple of $u \neq 0$ can be thought of as a nonzero element of $S/P \cong Su \subseteq M$, and so has annihilator P as well. Replace u by a nonzero multiple with as few nonzero homogeneous components as possible. If u_i is a nonzero homogeneous component of u of degree i, its annihilator J_i is easily seen to be a homogeneous ideal of S. If $J_h \neq J_i$ we can choose a form F in one and not the other, and then Fu is nonzero with fewer homgeneous components then u. Thus, the homogeneous ideals J_i are all equal to, say, J, and clearly $J \subseteq P$. Suppose that $s \in P - J$ and subtract off all components of s that are in J, so that no nonzero component is in J. Let $s_a \notin J$ be the lowest degree component of s and u_b be the lowest degree component in u. Then $s_a u_b$ is the only term of degree $a + b$ occurring in $su = 0$, and so must be 0. But then $s_a \in \mathrm{Ann}_S u_b = J_b = J$, a contradiction. □

Corollary. *Let K be a field and let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$. Let $\mathcal{M} = \bigoplus_{d=1}^{\infty} R_j$ be the homogeneous maximal ideal of R. Then* $\dim(R) = \mathrm{height}(\mathcal{M}) = \dim(R_{\mathcal{M}})$.

Proof. The dimension of R will be equal to the dimension of R/P for one of the minimal primes P of R. Since P is minimal, it is an associated prime and therefore is homogenous. Hence, $P \subseteq \mathcal{M}$. The domain R/P is finitely generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathcal{M}/P$. Thus,

$$\dim(R) = \dim(R/P) = \dim\big((R/P)_{\mathcal{M}}\big) \leq \dim R_{\mathcal{M}} \leq \dim(R),$$

and so equality holds throughout, as required. □

Proposition (homogeneous prime avoidance). *Let R be an $\mathbb{N}$-graded algebra, and let I be a homogeneous ideal of R whose homogeneous elements have positive degree. Let $P_1, \ldots, P_k$ be prime ideals of R. Suppose that every homogeneous element $f \in I$ is in $\bigcup_{i=1}^{k} P_i$. Then $I \subseteq P_j$ for some j, $1 \leq j \leq k$.*

Proof. We have that the set H of homogeneous elements of I is contained in $\bigcup_{i=1}^k P_k$. If $k = 1$ we can conclude that $I \subseteq P_1$. We use induction on k. Without loss of generality, we may assume that H is not contained in the union of any $k-1$ if the P_j. Hence, for every i there is a homogeous element $g_i \in I$ that is not in any of the P_j for $j \neq i$, and so it must be in P_i. We shall show that if $k > 1$ we have a contradiction. By raising the g_i to suitable positive powers we may assume that they all have the same degree. Then $g_1^{k-1} + g_2 \cdots g_k \in I$ is a homogeneous element of I that is not in any of the P_j: g_1 is not in P_j for $j > 1$ but is in P_1, and $g_2 \cdots g_k$ is in each of $P_2, \ldots, P_k$ but is not in P_1. □

We can now connect the dimension of a module with the degree of its Hilbert polynomial.

Theorem. *Let R be a polynomial ring $K[x_1, \ldots, x_n]$ over a field K, and let M be a finitely generated $\mathbb{Z}$-graded module over R. If M has dimension 0, the Hilbert polynomial of M is 0. If* $\dim(M) > 0$*, the Hilbert polynomial of M has degree* $\dim(M) - 1$.

Proof. M has dimension 0 if and only if it is killed by a power of $m = (x_1, \ldots, x_n)R$, in which case $[M]_d = 0$ for all $d \gg 0$. We use induction on $\dim(M)$.

If $\dim(M) > 0$, then exactly as in the Remark on p. 3 we may construct a finite filtration of M in which all the factors are prime cyclic modules, but using the fact that associated primes of graded modules are graded, we may assume that every R/P_i occurring is graded, i.e., that every P_i is homogeneous. Then the dimension of M is the same as the largest dimension of any R/P_i, and the degree of the Hilbert polynomial is the same as the largest degree of the Hilbert polynomial of any R/P_i. (The Hilbert polynomial of M is the sum of the Hilbert polynomials of the R/P_i. Note that we cannot have cancellation of leading coefficients in the highest degree because the leading coefficient of a Hilbert polynomial is positive: it cannot be negative, since the vector dimension of the space of forms $[R/P_i]_d$ for $d \gg 0$ cannot be negative.)

We have therefore reduced to the case where M has the form R/P, and has positive dimension. It follows that some x_i is not in P, and so there is a form f of degree 1 that is nonzero in the domain R/P. The dimension of $N = M/fM$ must be exactly $\dim(M) - 1$: the dimension must drop because we are killing a nonzero element in a domain, and it cannot drop by more than one, because the rings R/P and $R/(P + fR)$ have the same dimension when localized at their maximal ideals, and we may apply the Corollary at the top of p. 2.

We then have a short exact sequence of graded modules and degree preserving maps:

$$0 \to M(-1) \xrightarrow{f} M \to M/fM \to 0,$$

so that if H_M denotes the Hilbert polynmial of M we have so that

$$(*) \quad H_M(d) - H_M(d-1) = H_{M/fM}(d)$$

for all d. In general, if $P(d)$ is a polynomial in d of degree $k \geq 1$ and with leading coefficient a, the *first difference* $P(d) - P(d-1)$ is a polynomial of degree $k-1$ with leading coefficient

ka. Therefore, the degree of the left hand side is $\deg(H_M) - 1$, while the right hand side, by the induction hypothesis, is a polynomial of degree $\dim(M/fM) - 1$ (if $\dim(M) > 1$) or is 0 (if $\dim(M) = 1$). Since $\dim(M/fM) = \dim(M) - 1$, the result follows. $\square$

We saw in the final Theorem of the Lecture Notes of January 22 that F/M and $F/\mathrm{in}(M)$ have the same Hilbert-Poincaré series when M is a graded submodule of a finitely generated free module over a polynomial ring $R = K[x_1, \ldots, x_n]$. Of course, this also means that F/M and $F/\mathrm{in}(M)$ have the same Hilbert function and, hence, the same Hilbert polynomial. We therefore can reduce the problem of finding the Krull dimension of a module to the monomial case:

Theorem. *Let N be any finitely generated $\mathbb{Z}$-graded module over $R = K[x_1, \ldots, x_n]$. Suppose that $u_1, \ldots, u_s$ are finitely many homogeneous generators of respective degrees $d_1, \ldots, d_s$. Think of R^s as $\bigoplus_{j=1}^s R(-d_j)$, and map $R^s \twoheadrightarrow N$ so that $1 \in R(-d_j)$, which has degree d_j, maps to u_j. This map preserves degrees, and the kernel M is an $\mathbb{N}$-graded submodule of R^s.*

Refine the $\mathbb{Z}$-grading on R^s to a $\mathbb{Z}^n$-grading, and choose a monomial order. Then $\dim(N) = \dim\big(F/\mathrm{in}(M)\big)$. $\square$

Since a monomial submodule M of F is a direct sum $I_1e_1 \oplus \cdots I_se_s$, where every I_j is a monomial ideal, we have that $\dim(F/M) = \sup_j\{\dim(R/I_j)\}$. We have therefore reduced the problem of finding the dimension of a module M to that of finding the dimension of R/I when I is a monomial ideal. We can make one more simplification: since $R/\mathrm{Rad}\,(I)$ and R/I have the same dimension, it suffices to consider the case where I is a radical ideal generated by monomials. Since $(x_{i_1} \cdots x_{i_h})^k$ is a multiple of $x_{i_1}^{a_1} \cdots x_{i_h}^{a_h}$ (here, the a_i are positive integers) whenever $k \geq \sup_j a_j$, the radical of an ideal generated by monomials is generated by square-free monomials. (It is easy to check that any ideal generated by square-free monomials in $K[x_1, \ldots, x_n]$ is, in fact, radical.)

Rings defined by killing square-free monomials and simplicial complexes

By a finite simplicial complex Σ with vertices $x_1, \ldots, x_n$ we mean a set of subsets of $\{x_1, \ldots, x_n\}$ such that

(1) For $1 \leq i \leq n$, $\{x_i\} \in \Sigma$.

(2) Every subset of a set in Σ is also in Σ.

The sets $\sigma \in \Sigma$ are called the *faces*. The *dimension* of σ is one less than its cardinality: the elements of Σ of dimension i are called *i-simplices* of σ. The *dimension* of Σ is the largest dimension of any face. The maximal faces of Σ are called *facets* and these determine Σ: a set is in Σ if and only if it is a subset of a facet of Σ.

If we think of $x_1, \dots, x_n$ as the points $e_1, \dots, e_n$ in $\mathbb{R}^n$, where e_i has 1 in the i th spot and 0 elsewhere, we can define *the geometric realization* $|\Sigma|$ of Σ to be the topological space

$$\bigcup_{\sigma \in \Sigma} \text{convex hull}(\sigma)$$

in $\mathbb{R}^n$. The dimension of Σ then coincides with its dimension as a topological space.

Example. If Σ has three vertices x_1, x_2, x_3 and facets $\{x_1, x_2\}$, $\{x_1, x_3\}$, and $\{x_2, x_3\}$, then $|\Sigma|$ is the union of three line segments: it is a triangle, without the interior. On the other hand, if Σ has one facet, $\{x_1, x_2, x_3\}$, then $|\Sigma|$ is a triangle with interior.

Our reason for discussing simplicial complexes at this point is that there is a bijective correspondence between the square-free monomial ideals in $K[x_1, \dots, x_n]$ that do not contain any of the variables $x_1, \dots, x_n$ and the simplicial complexes with vertices $x_1, \dots, x_n$. One may let the ideal I correspond to the subsets of $\{x_1, \dots, x_n\}$ whose product is *not in* I. Notice that if a monomial ideal does contain one of the variables x_i, the quotient R/I may be thought of as a quotient of a polynomial ring in fewer variables (omitting x_i) by square-free monomials.

The ring R/I_Σ corresponding to simplicial complex Σ is called the *face* ring or *Stanley-Reisner* ring of Σ over K. Here, I_Σ is simply the ideal generated by all square-free monomials such that the set of variables occurring is not a face of Σ.

We leave it as an exercise to verify the minimal primes of R/I_Σ correspond bijectively to the facets of Σ: each minimal prime Q is generated by the images of the elements in $\{x_1, \dots, x_n\} - \sigma$ for some facet σ, the quotient by Q is the isomorphic to a polynomial ring in the variables that occur in σ. It then follows that $\dim(R/I_\Sigma) = \dim(\Sigma) + 1$.

Elimination theory

We now return to the problem of finding the intersection of an ideal $I \subseteq K[x_1, \dots, x_n]$ with $K[x_{k+1}, \dots, x_n]$, which also gives an algorithm for solving a finite system of polynomial equations over an algebraically closed field when there are only finitely many solutions. The method is incredibly simple!

Theorem. *If $g_1, \dots, g_r$ is a Gröbner basis for I with respect to lexicographic order, then the elements of this basis that lie in $K[x_{k+1}, \dots, x_n]$ are a Gröbner basis for the ideal $J = I \cap K[x_{k+1}, \dots, x_n]$.*

Proof. Let $g_{h+1}, \dots, g_r$ be the elements of the Gröbner basis that lie in $K[x_{k+1}, \dots, \boldsymbol{x}_n]$ (if $g_1, \dots, g_r$ are in order of the sizes of their initial terms, these elements will be consecutive and at the end of the sequence).

Consider any element $f \in J$. Then there is a standard expression for f divided by $g_1, \dots, g_r$, and the remainder will be zero. Say the expression is $f = \sum_{j=1}^{n} q_j g_j$. Any g_j

that involves one of $x_1, \dots, x_k$ has initial term involving one of the variables $x_1, \dots, x_k$, and the initial term of $q_j g_j$ will be too large to use in the standard expression unless $q_j = 0$. Therefore, we actually have $f = \sum_{j=h+1}^{n} q_j g_j$. The same reasoning shows that any q_j for $j > k$ involves only $x_{k+1}, \dots, x_n$. The initial term of f must be the same, up to a nonzero scalar multiple, as the intitial term of one of the $q_j g_j$, and so it is in the $K[x_{k+1}, \dots, x_n]$-span of $g_{h+1}, \dots, g_r$. $\square$