Math 615: Lecture of January 29, 2007

We next want to discuss the notion of a *regular sequence* in a ring or on a module. We are aiming to discuss criteria, using revlex, for a sequence to be regular on F/M. However, we also want to discuss some theorems that we are aiming to prove eventually about the Cohen-Macaulay property for $\mathbb{N}$-graded algebras finitely generated over a field K.

A sequence of elements $f_1, \ldots, f_k \in R$, where R is a ring, is said to a *regular sequence* on the R-module M (when $M = R$, one may refer to a *regular sequence on* R or a *a regular sequence in* R if

(1) $(f_1, \ldots, f_n)M \neq M$,

(2) f_1 is not a zerodivisor on M, i.e., $M \xrightarrow{f_1 \cdot} M$ is injective.

(3) For all i, $1 \leq i \leq k-1$, f_{i+1} is not a zerodivisor on $M/(f_1, \ldots, f_k)M$.

These conditions can be expressed more concisely by allowing $i = 0$ in condition (1), with the interpretation that $(f_1, \ldots, f_i)M = 0$ if $i = 0$.

The empty sequence is regular sequence on every nonzero module M.

Condition (1) is assumed in order to eliminate certain degerate situations. Without it, the sequence 1, 1, ,1, ... , 1 (of any desired length) would be a regular sequence on the 0 module, for example.

Note that $f_1, \ldots, f_h, f_{h+1}, \ldots, f_k$ is a regular sequence on M if and only if $f_1, \ldots, f_h$ is a regular sequence on M and $f_{h+1}, \ldots, f_k$ is a regular sequence on $M/(f_1, \ldots, f_h)M$.

The term *Rees sequence* on M is also used, as well as the term R*-sequence* on M (where "R" may be thought of as standing for "Rees" or "regular"). The term M*-sequence* is also used. We shall always use the term "regular sequence," however.

For example, if $x_1, \ldots, x_n$ are indeterminates, $x_1, \ldots, x_n$ is a regular sequence on $R = K[x_1, \ldots, x_n]$ and on $S = K[[x_1, \ldots, x_n]]$, as well as on any free R-module or free S-module. In fact, we will show that a finitely generated S-module (respectively, a finitely generated $\mathbb{Z}$-graded R-module) M is S-free (respectively, R-free) if and only if $x_1, \ldots, x_n$ is a regular sequence on M.

It is worth noting that, in general, regular sequences are not permutable, even in very well-behaved rings. For example, in the polynomial ring $R = K[x, y, z]$, x, $(1-x)y$, $(1-x)z$ is a regular sequence, but $(1-x)y$, $(1-x)z$, x is not. For the former, modulo xR, the latter two elements become y and z in $K[y, z]$. For the second sequence, $(1-x)z$ is a zerodivisor modulo $(1-x)yR$: the image of y is not 0, but $(1-x)z$ kills the image of y. However, we shall see that regular sequences are permutable in the local case when the module is finitely generated, and in certain graded cases (a precise statement is given below).

Before considering properties of regular sequences further, we want to discuss the local and graded versions of Nakayama's Lemma.

Nakayama's Lemma. *Let R be a ring and let M be an R-module. Suppose that either of the followng two conditions holds:*

(1) *R has a unique maximal ideal m and M is finitely generated.*

(2) *R is $\mathbb{N}$-graded, $m \subseteq R$ consists entirely of elements whose homogeneous components have positive degree, and M is $\mathbb{Z}$-graded, but $[M]_{-d} = 0$ for all $d \gg 0$.*

If $mM = M$ then $M = 0$.

Proof. In case (1) let $u_1, \ldots, u_k$ be a set of generators of M of smallest cardinality. If $k = 0$ then $M = 0$ and we are done. If not, then $u_k \in mM = m(Ru_1 + \cdots + Ru_k) = mu_1 + \cdots + mu_k$, and so $u_k = f_1u_1 + \cdots + f_ku_k$ with every $f_j \in m$. Then $(1 - f_k)u_k = f_1u_1 + \cdots + f_{k-1}u_{k-1}$, and $1 - f_1 \notin m$. It follows that $1 - f_1$ is a unit of R. If $g = (1 - f_k)^{-1}$, then $u_k = gf_1u_1 + \cdots + gf_{k-1}u_{k-1}$, and $u_1, \ldots, u_{k-1}$ generate M, contradicting the minimality of k.

In case (2), let $u \in M$ be a nonzero homogeneous element of smallest possible degree. Then $u \in mM$ implies that u is a sum of elements f_jv_j where the f_j are homogeneous of positive degree and the v_j are homogeneous. Then u is the sum of those nonzero terms f_jv_j such that $\deg(f_j) + \deg(v_j) = \deg(u)$. For those v_j occurring, this implies that $\deg(v_j) = \deg(u) - \deg(f_j) < \deg(u)$, a contradiction. □

Corollary. *Let R be a ring and let M be an R-module. Suppose that either of the followng two conditions holds:*

(1) *R has a unique maximal ideal m and M is finitely generated.*

(2) *R is $\mathbb{N}$-graded, $m \subseteq R$ consists entirely of elements whose homogeneous components have positive degree, and M is $\mathbb{Z}$-graded, but $[M]_{-d} = 0$ for all $d \gg 0$.*

If the images of the elements $\{u_\lambda\}_{\lambda \in \Lambda}$ generate M/mM (and, in case (2), are homogeneous) then the elements $\{u_\lambda\}_{\lambda \in \Lambda}$ generate M.

Proof. Let N be the R-span of $\{u_\lambda\}_{\lambda \in \Lambda}$. In case (2), N and M/N are homogeneous. Since the images of the u_λ span M/mM, we have that $N + mM = M$, and consequently we also have that $(mM + N)/N = M/N$, and this implies that $m(M/N) = M/N$. Thus, by the appropriate case of Nakayama's Lemma, $M/N = 0$, and $M = N$. □

As a consequence of Nakayama's Lemma, we can prove the permutability of regular sequences in local and graded cases.

Proposition (permutability of regular sequences). *Let R be a ring, let M be an R-module, and let $f_1, \ldots, f_k \in R$ be a regular sequence on M. Suppose that either of the following two conditions holds:*

(1) *(R, m, K) is local, $f_1, \ldots, f_k \in m$, and M is finitely generated.*

(2) *R is $\mathbb{N}$-graded, M is $\mathbb{Z}$-graded but $[M]_{-d} = 0$ for all $d \gg 0$, and $f_1, \ldots, f_k$ are homogeneous of positive degree.*

For every permutation π of $1, 2, \ldots, k$, $f_{\pi(1)}, f_{\pi(2)}, \ldots, f_{\pi(k)}$ is a regular sequence on M.

Proof. Because the permutations on $1, 2, \ldots, k$ are generated by transpositions $(i\ i+1)$ of consecutive integers, we need only consider the case where π is such a transposition. We may replace M by $M/(f_1, \ldots, f_{i-1})M$ without affecting any relevant issues. Thus, we may assume without loss of generality that we are simply transposing the first two terms of the regular sequence. But once we have shown that f_2, f_1 is a regular sequence, the rest is automatic, since $M/(f_1, f_2)M = M/(f_2, f_1)M$. Therefore, we need only consider the case where $k = 2$ and we are transposing the elements.

We first need to see that f_2 is not a zerodivisor on M. Let $N \subseteq M$ be the annihilator of f_2. (In the graded case, N is graded.) If $u \in N$, then $f_2u = 0$ certainly implies that $f_2u = f_1v$, and so $u = f_1w$ for some $w \in M$. But then $0 = f_2u = f_2f_1w = f_1(f_2w)$, and since f_1 is not a zerodivisor on M, we have that $f_2w = 0$, so that $w \in N$. But we have now shown that if $u \in N$, then $u = f_1w$ with $w \in N$. Thus, $N = f_1N$. By the appropriate form of Nakayama's Lemma, $N = 0$.

Now suppose that $f_1v = f_2u$ where $v, u \in M$, so that f_1 kills the image of v in M/f_2M. Then, since f_2 is not a zerodivisor on M/f_1M, we have that $u \in f_1M$, say $u = f_1w$. Then $f_1v = f_2f_1w$ and $f_1(v - f_2w) = 0$. Since f_1 is not a zerodivisor on M, $v = f_2w$. $\square$

Regular local rings

A local ring (R, m, K) is called *regular* if the Krull dimension of R is equal to the least number of generators of the maximal ideal m. The least number of generators of m is the K-vector space dimension of m/m^2 by Nakayama's Lemma: $\dim_K(m/m^2)$ is called the *embedding dimension* of R. The Krull dimension is the least number of generators of an ideal whose radical is m, and we always have $\dim(R) \leq \dim_K(m/m^2)$.

If $\dim(R) = 0$, R is regular if and only if R is a field.

If $\dim(R) = 1$, then m is generated by one element x, which is not nilpotent. Every nonzero element can be written as a unit times a power of x, since the intersection of the powers of m is 0: simply factor out x as many times as possible. It follows that R is a domain. Thus, the one dimensional regular local rings are precisely the Noetherian discrete valuation rings: we refer to such a ring briefly as a DVR.

Higher dimensional examples include formal power series rings over a field or a DVR.

Note that if R is regular and $x_1, \ldots, x_k$ have images that are linearly indepedent in m/m^2, then $\overline{R} = R/(x_1, \ldots, x_k)R$ is again regular. (Call the maximal ideal in the quotient ring $\overline{m}$. We can extend the sequence to $x_1, \ldots, x_n$, where $n = \dim(R)$, and then the images of the remaining elements $x_{k+1}, \ldots, x_n$ are linearly independent in $\overline{m}/\overline{m}^2$ and are a system of parameters for $\overline{R}$).

We have:

Theorem. *A regular local ring (R, m, K) is a domain, and a local ring is regular if and only if its maximal ideal m is generated by a regular sequence.*

Proof. We use induction on $\dim(R)$ to prove that R is a domain. Therefore, we may assume that $\dim(R) \geq 2$. Let x, y have linearly independent images in m/m^2. It follows that each of the elements $x - y^n$ is prime, for $R/(x - y^n)$ is a regular, and is a domain by the induction hypothesis. It is easy to see that none of these elements divides any of the others. If $x - y^n$ were a multiple of $x - y^h$ then in $R/(x - y^h)$ the images of $x - y^n$ and $x - y^h$ are both 0, and so $y^n \equiv y^h$. Since $R/(x - y^h)$ is a domain, this forces $y \equiv 0$ (y is in the maximal ideal, and so no power of y can equal 1). But then $(x - y^h)R$ contains y, which is false even modulo m^2. If $uv = 0$ in R, then either u is divisible by infinitely many $x - y^n$ or v is. Suppose u is. But the intersection of ideals generated by prime elements, none of which divides any of the others, is their product. This forces u into arbitrarily high powers of m, and so $u = 0$.

It now follows that if $x_1, \ldots, x_n$ generate m minimally, then $R/(x_1, \ldots, x_k)$ is a domain for every k, and so x_{k+1} is not a zerodivisor modulo $(x_1, \ldots, x_k)R$.

On the other hand, if m is generated by a regular sequence one sees at once that the dimension and embedding dimension of R are the same. $\square$

We can now charactize when a module is free in terms of regular sequences in certain cases. We need Nakayama's Lemma to hold.

Theorem. *Let R be a ring and $M \neq 0$ an R-module. Suppose that one of the following conditions holds:*

(1) *(R, m, K) is regular local, $x_1, \ldots, x_n$ is a regular sequence generating m, and M is finitely generated.*

(2) *$R = K[x_1, \ldots, x_n]$ is a polynomial ring over a field K, and M is $\mathbb{Z}$-graded such that $[M]_{-d} = 0$ for all $d \gg 0$. Then M is free if and only if $x_1, \ldots, x_n$ is a regular sequence on M.*

Proof. In both cases, $x_1, \ldots, x_n$ form a regular sequence on R. If elements form a direct sequence on each module in a family, then they form a regular sequence on the direct sum. Hence, $x_1, \ldots, x_n$ is a regular sequence on any free module.

It remains to show that, under the hypothesis of the Theorem, if $x_1, \ldots, x_n$ form a regular sequence on M then M is free. Choose elements $\{u_\lambda\}_{\lambda \in \Lambda}$ in M whose images are a K-vector space basis for $M/(x_1, \ldots, x_n)M$. Moreover, in case (2) choose these elements to be homogeneous. By the appropriate form of Nakayama's Lemma, they span M. It is therefore sufficient to prove that they are independent over R. We use induction on n. The case $n = 0$ is clear. Assume that $n \geq 1$. It follows that M/x_1M is free on the images of the u_λ over R/x_1R. Consider h elements from this set of generators, say $u_1, \ldots, u_h$, and let

$N \subseteq R^h$ be the set of relations on these elements over R. (In the graded case, let u_i have degree s_i and view R_h as $R(-s_1) \oplus \cdots \oplus R(-s_h)$.) In the graded case, N is graded. We can complete the proof by showing that $N = 0$. Now consider any relation $(f_1, \ldots, f_h)$ on $u_1, \ldots, u_h$, so that $f_1u_1 + \cdots + f_hu_h = 0$. Working modulo x_1M (and x_1R), we see that we must have that every f_j is divisible by x_1, say $f_j = x_1g_j$. Then $x_1(g_1u_1 + \cdots g_hu_h) = 0$, and x_1 is not a zerodivisor on M. It follows that $(g_1, \ldots, g_h) \in N$. Thus, $N = x_1N$. By the appropriate form of Nakayama's Lemma, $N = 0$. $\square$

Discussion: homogeneous systems of parameters. Let R be a finitely generated $\mathbb{N}$-graded K-algebra, where $R_0 = K$. Let $m = \bigoplus_{d=1}^{\infty} R_d$ be the homogeneous maximal ideal of R. Since the minimal primes of R are homogeneous, if $\dim(R) > 0$ we can choose a form $F_1 \in m$ such that F_1 is not in any minimal prime of R. Then $\dim(R/F_1R) = \dim(R) - 1$. Now suppose that forms $F_1, \ldots, F_i$ have been chosen such that $\dim\bigl(R/(F_1, \ldots, F_i)R\bigr) = \dim(R) - i$. If $i < n = \dim(R)$, we can choose $F_{i+1} \in m$ not in any minimal prime (these are homogeneous) of $(F_1, \ldots, F_i)R$, and it follows that $\dim\bigl(R/(F_1, \ldots, F_{i+1})\bigr) = \dim(R) - (i+1)$. Thus, eventually we have a sequence of forms $F_1, \ldots, F_n$ of positive degree such that $\dim\bigl(R/(F_1, \ldots, F_n)\bigr) = 0$. Such a sequence of forms is called a *homogeous system of parameters* for R.

Theorem. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$ such that $\dim(R) = n$. A homogeneous system of parameters $F_1, \ldots, F_n$ for R always exists. Moreover, if $F_1, \ldots, F_n$ is a sequence of homogeneous elements of positive degree, then the following statements are equivalent.*

(1) *$F_1, \ldots, F_n$ is a homogeneous system of parameters.*

(2) *m is nilpotent modulo $(F_1, \ldots, F_n)R$.*

(3) *$R/(F_1, \ldots, F_n)R$ is finite-dimensional as a K-vector space.*

(4) *R is module-finite over the subring $K[F_1, \ldots, F_n]$.*

Moreover, when these conditions hold, $F_1, \ldots, F_n$ are algebraically independent over K, so that $K[F_1, \ldots, F_n]$ is a polynomial ring.

Proof. We have already shown existence.

(1) $\Rightarrow$ (2). If $F_1, \ldots, F_n$ is a homogeneous system of parameters, we have that $\dim\bigl(R/F_1, \ldots, F_n)\bigr) = 0$. We then know that all prime ideals are maximal. But we also know that the maximal ideals are also minimal primes, and so must be homogeneous. Since there is only one homogenous maximal ideal, it must be $m/(F_1, \ldots, F_n)R$, and so m is nilpotent on $(F_1, \ldots, F_n)R$.

(2) $\Rightarrow$ (3). If m is nilpotent modulo $(F_1, \ldots, F_n)R$, then the homogeneous maximal ideal of $\overline{R} = R/(F_1, \ldots, F_n)R$ is nilpotent, and it follows that $[\overline{R}]_d = 0$ for all $d \gg 0$. Since each $\overline{R}_d$ is a finite dimensional vector space over K, it follows that $\overline{R}$ itself is finite-dimensional as a K-vector space.

(3) $\Rightarrow$ (4). This is immediate from the homogeneous form of Nakayama's Lemma: a finite set of homogeneous elements of R whose images in $\overline{R}$ are a K-vector space basis

will span R over $K[F_1, \ldots, F_n]$, since the homogenous maximal ideal of $K[F_1, \ldots, F_n]$ is generated by $F_1, \ldots, F_n$.

(4) $\Rightarrow$ (1). If R is module-finite over $K[F_1, \ldots, F_n]$, this is preserved mod $(F_1, \ldots, F_n)$, so that $R/(F_1, \ldots, F_n)$ is module-finite over K, and therefore zero-dimensional as a ring.

Finally, when R is a module-finite extension of $K[F_1, \ldots, F_n]$, the two rings have the same dimension. Since $K[F_1, \ldots, F_n]$ has dimension n, the elements $F_1, \ldots, F_n$ must be algebraically independent. $\square$

Discussion: making a transition from one system of parameters to another. Let R be a Noetherian ring of Krull dimension n, and assume that either

(1) (R, m, K) is local and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two systems of parameters.

(2) R is finitely generated $\mathbb{N}$-graded over $R_0 = K$, a field, m is the homogeneous maximal ideal, and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two homogeneous systems of parameters for R.

We want to observe that in this situation there is a finite sequence of systems of parameters (respectively, homogeneous systems of parameters in case (2)) starting with $f_1, \ldots, f_n$ and ending with $g_1, \ldots, g_n$ such that any two consecutive elements of the sequence agree in all but one element (e.g., after reordering, only the i th terms are possibly different for a single value of i, $1 \leq i \leq n$. We can see this by induction on n. If $n = 1$ there is nothing to prove. If $n > 1$, first note that we can choose h (homogeneous of positive degree in the graded case) so as to avoid all minimal primes of $(f_2, \ldots, f_n)R$ and all minimal primes of $(g_2, \ldots, g_n)R$. Then it suffices to get a sequence from $h, f_2, \ldots, f_n$ to $h, g_2, \ldots, g_n$, since the former differs from $f_1, \ldots, f_n$ in only one term and the latter differs from $g_1, \ldots, g_n$ in only one term. But this problem can be solved by working in R/hR and getting a sequence from the images of $f_2, \ldots, f_n$ to the images of $g_2, \ldots, g_n$, which we can do by the induction hypothesis. We lift all of the systems of parameters back to R by taking, for each one, h and inverse images of the elements in the sequence in R (taking a homogeneous inverse image in the graded case), and always taking the same inverse image for each element of R/hR that occurs. $\square$

Cohen-Macaulay rings were discussed in the first lecture. But we are now in a position to prove several of the assertions made there.

Theorem. *Let R be a finitely generated graded algebra over $R_0 = K$. The following conditions are equivalent.*

(1) *Some homogeneous system of parameters is a regular sequence.*

(2) *Every homogeneous system of parameters is a regular sequence.*

(3) *For some homogeneous system of parameters $F_1, \ldots, F_n$, R is a free-module over $K[F_1, \ldots, F_n]$.*

(4) *For every homogeneous system of parameters $F_1, \ldots, F_n$, R is a free-module over $K[F_1, \ldots, F_n]$.*

Proof. We first show that (1) and (2) are equivalent. We want to show that if one homogeneous system of parameters is a regular sequence, then every homogeneous system of parameters is a regular sequence. By the Discussion above, we may assume that they agree except possibly in one term. Since regular sequences are permutable (and systems of parameters are obviously permutable), we may assume that they agree except possibly for the last term. Call them $F_1, \ldots, F_n$ and $F_1, \ldots, F_{n-1}, G$. The issue is whether the last term is a nonzerodivisor modulo the earlier terms. Therefore, we may pass to $\overline{R} = R/(F_1, \ldots, F_{n-1})$, which is one-dimensional. It follows that we may assume that R is one-dimensional, and we need only show that if F, G both generate ideals whose radical is m and F is a nonzerodivisor, then G is a nonzerodivisor. But F has a power in GR, say $F^k = GH$. If G is a zerodivisor, it follows that F^k is as well, and then F must be a zerodivisor. This proves the equivalence of (1) and (2). The preceding Theorem yields the equivalence of (1) and (3), as well as the equivalence of (2) and (4), immediately. $\square$

As mentioned earlier, we shall say that R is *Cohen-Macaulay* of these equivalent conditions hold. The same argument as given in the proof just above also shows:

Theorem. *Let (R, m, K) be a local ring. Then one system of parameters is a regular sequence if and only if every system of parameters is a regular sequence.* $\square$

We shall say that the local ring R is *Cohen-Macaulay* if every system of parameters is a regular sequence. Of course, regular rings are Cohen-Macaulay. We shall later show that an $\mathbb{N}$-graded ring over $R_0 = K$ is Cohen-Macaulay if and only if all of its local rings are Cohen-Macaulay.

We shall eventually prove two substantial results about when rings are Cohen-Macaulay. One of them is Reisner's criterion for when the face ring of a finite simplicial complex is Cohen-Macaulay. The other concerns the Cohen-Macaulay property for certain rings of invariants of matrix groups acting on polynomial rings.

To state Reisner's criterion, we need the notion of *link* in a simplicial complex Σ. If x is a vertex of Σ, we define the *link* of x in Σ to be the simplicial complex Λ such that $\tau \in \Lambda$ if and only if $\tau \in \Sigma$, $x \notin \tau$, and $\{x\} \cup \tau \in \Sigma$.

For example, suppose that Σ corresponds to the triangulation of a convex pentagon obtained by connecting an interior point to the vertices, and x is the interior point. If the vertices on the perimeter are $x_1, \ , x_2, x_3, x_4, x_5$, then the facets of Σ are the five 2-simples $\{x, x_i, x_{i+1}\}$, for $1 \leq i \leq 5$, where x_{i+1} is to be interpreted as x_1 when $i = 5$ (i.e., the subscripts are read modulo 5). The link of x is the perimeter of the pentagon: its facets are the five 1-simplices (or *edges*) $\{x_i, x_{i+1}\}$, where $1 \leq i \leq 5$.

If we take Σ to have facets $\{x_1, x_3\}$, $\{x_2, x_3\}$, and $\{x_3, x_4, x_5\}$ (the geometric realization consists of a triangle with interior and two additional line segments jutting out from one vertex), then the link of x_3 has facets $\{x_1\}$, $\{x_2\}$, and $\{x_4, x_5\}$: a line segment with two additional isolated points.

Once one has a link, one can treat it as a new simplicial complex, and take the link of one of its vertices. This may be iterated several times. But these iterated links can be obtained in a single step as follows. If $\sigma_0 \in \Sigma$, define the *link* of $\sigma_0 \in \Sigma$ as the simplical complex $\{\tau \in \Sigma : \tau \cap \sigma_0 = \emptyset \text{ and } \tau \cup \sigma_0 \in \Sigma\}$. One gets the same simplicial complex by iterating the operation of taking links of vertices, using all vetices in σ_0: the iterated link obtained is independent of the order in which one takes links of vertices.

We also recall that the *reduced simplicial homology* of Σ over K is the the same as the simplicial homology over K, except in dimension 0, where it has K-vector space dimension one smaller. (Thus $\widetilde{H}_0(X;\,K) = 0$ if and only if Σ is connected.)

We can now state:

Theorem (Reisner). *Let K be a field, let Σ be a finite simplical complex with vertices $x_1, \ldots, x_n$, and let I_Σ be the ideal of $R = K[x_1, \ldots, x_n]$ generated by the square free monomials such that the set of variables that occur is not a face of Σ. Then R/I_Σ is Cohen-Macaulay if and only if both of the following conditions hold:*

(1) *The reduced simplicial homology $\widetilde{H}_i(\Sigma;\,K)$ with coefficients in K vanishes, $0 \le i \le \dim(\Sigma) - 1$.*

(2) *For every link Λ, the reduced simplicial homology $\widetilde{H}_i(\Lambda;\,K) = 0$, $0 \le i \le \dim(\Lambda) - 1$.*

We defer the proof. We also note that by a result of Munkres, Reisner's condition is actually a topological property of $|\Sigma|$.

Note that in dimension 0, every finite simplicial complex is Cohen-Macaulay. In dimension 1, Σ is Cohen-Macaulay if and only if it is connected.

In dimension 2, a triangulation of a sphere gives a Cohen-Macaulay ring, a triangulation of a cylinder does not, while what happens with a triangulation of a real projective plane depends on the characteristic. In characteristic 2, the first homology group of the the real projective plane does not vanish, and the ring one gets is not Cohen-Macaulay. In all other characteristics, the ring is Cohen-Macaulay.

Finally, we mention one more Theorem. Let G be a Zariski closed subgroup of $\mathrm{GL}(n,\,K)$: thus, G is a group of matrices. Suppose that G is linearly reductive, by which we mean that every (algebraic) representation is completely reducible. There are many such groups in characteristic 0: the general and special linear groups, the orthogonal group, and the symplectic group are examples, as well as finite groups, the multiplicative group of the field, and products of the groups already mentioned In characteristic $p > 0$, there are relatively few such groups: products of copies of the multiplicative group of the field and finite groups whose order is not divisibile by p are the main examples .

Then G may be thought of as acting on the space of forms of degree 1 in $K[x_1, \ldots, x_n]$, and the action extends to an action on the polynomial ring R itself. One may form the ring of invariants $R^G = \{f \in R : \gamma(f) = f \text{ for all } \gamma \in G\}$. When G is linearly reductive, this group turns out to be finitely generated. Beyond that:

Theorem. *With hypothesis as in the paragraph above, R^G is a Cohen-Macaulay ring.*