Math 615: Lecture of January 31, 2007

Invariant Theory

We want to present some examples from classical invariant theory to which one can apply the Theorem on the Cohen-Macaulay property for rings of invariants stated at the end of the Lecture Notes of January 29, as well as a strong form, stated below.

For simplicity, in this discussion we assume that we are working over an algebraically closed field K when describing what is meant by a linear algebraic group and an action of such a group: this minimizes prerequisites from algebraic geometry. However, the statements identifying the rings of invariants of various group actions are all valid over any infinite field, and the statements about rings being Cohen-Macaulay are valid over any field. In fact, we note the following result:

Proposition. *If R is a finitely generated graded K-algebra over a field K with $R_0 = K$, then R is Cohen-Macaulay if and only if $L \otimes_K R$ is Cohen-Macaulay.*

The proof is left as an exercise: see problem 4(d). of Problem Set #2.

Next note that if $X \subseteq \mathbb{A}^s_K$ is a closed algebraic set and $f \in K[X]$ is a regular function on X, the open subset $X_f = X - V(f)$ has the structure of a closed algebraic set embedded in $\mathbb{A}^{s+1}_K$: if $X = V(I)$, then X_f is in bijective correspondence with $V(I, fx_{s+1} - 1) \subseteq \mathbb{A}^{s+1}_K$. The coordinate ring of X_f is easily shown to be $K[X]_f$, and the inclusion $X_f \subseteq X$ corresponds to the natural K-algebra homomorphism $K[X] \to K[X]_f$.

Therefore, if we identify $n \times n$ matrices over K with $\mathbb{A}^{n^2}$ and D denotes the determinant function, $\mathrm{GL}(n, K)$ may be identified with $\mathbb{A}^{n^2}_D$, and so has the structure of a closed algebraic set. For any finite-dimensional vector space V over K, by choosing a basis we may identify the group $\mathrm{GL}_K(V)$ of K-linear automorphisms of V with $\mathrm{GL}(r, K)$, where $r = \dim(V)$, and so $\mathrm{GL}(V)$ acquires the structure of an algebraic set. Since conjugation by a fixed invertible $r \times r$ matrix is an automorphism of $\mathrm{GL}(r, K)$ as an algebraic set, the algebraic set structure on $GL(V)$ is independent of the choice of the K-vector space basis for V.

By a *representation* of the linear algebraic group G we mean a group homomorphism $G \to \mathrm{GL}_K(V)$ that is also a K-regular map of closed algebraic sets. The representation evidently gives an action of G on V, and may also be described by giving a K-regular map $G \times V \to V$ satisfying the conditions for a group action. A representation is called *irreducible* if no proper nonzero subspace W of V is stable under the action of V.

As was mentioned in the Lecture of January 29, G is called *linearly reductive* if every representation is *completely reducible*, which means that it is a direct sum of irreducible representations. As was also mentioned in that lecture, the general linear group and the special linear group are examples in characteristic 0. The multiplciative group of the field is $\mathrm{GL}(1, K)$: finite products of copies of the multiplicative group of the field (such groups are called *algebraic tori*) are examples in all characteristics.

In fact, one has the following more general statement:

Theorem. *Let G be a linearly reductive linear algebraic group over K acting on the vector space of forms of degree one in the polynomial ring $R = K[x_1, \ldots, x_n]$. The action extends uniquely to an action of G on R by degree-preserving K-algebra automorphisms. For this action, the ring of invariants R^G is Cohen-Macaulay.*

The proof is deferred for a while. One of the surprising aspects of this Theorem is that the most interesting examples are in characteristic 0, but the first proof [M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974) 115–175] of the result and, by far, the simplest proof [M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, Amer. J. Math. **3** (1990) 31–116] use reduction to characteristic $p > 0$.

In classical invariant theory there were two fundamental problems. The first was to determine generators for the ring of invariants of a group action. The second was to give generators for the ideal of relations on these generating invariants. See [Hermann Weyl, *The Classical Groups*, Princeton Univ. Press, Princeton, 1946] for the solution of several important problems of this type. In the light of the Theorem above, the rings of invariants studied classically provide many interesting examples of Cohen-Macaulay rings.

We want to consider some of these examples. We first introduce two notations. If X is a matrix with entries in a K-algebra R, we denote by $I_t(X)$ the ideal of R generated by the $t \times t$ minors (determinants of $t \times t$ submatrices) of X, and by $K[X]$ the K-subalgebra of R generated by the entries of X. More generally, we denote by $K[X/t]$ the K-subalgebra of R generated by the $t \times t$ minors of X.

In the three examples just below, the field is assumed *to be infinite.*

First example. Let $G = K - \{0\} \cong GL(1, K)$ act on the polynomial ring

$$R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

in $m + n$ variables so that if $a \in G$, $x_i \mapsto x_j a^{-1}$ and $y_j \mapsto a y_j$ for all i and j. It is easy to verify that the ring of invariants is

$$K[x_i y_j : 1 \le i \le m,\, 1 \le j \le n].$$

(It is certainly clear that these elements are invariant: $x_i a^{-1} a y_j = x_i y_j$.) If $U = \big(u_{ij}\big)$ is an $m \times n$ matrix of new indeterminates, we can map $K[U] \twoheadrightarrow R^G = K[x_i y_j : i, j]$ as K-algebras by sending $u_{ij} \mapsto x_i y_j$. Note that

$$(x_i y_j)(x_{i'} y_{j'}) = (x_i y_{j'})(x_{i'} y_j),$$

which shows that $I_2(U)$ is in the kernel. In fact, $R^G \cong K[U]/I_2(U)$. This ring is Cohen-Macaulay in all characteristics.

Second example. We can generalize the preceding example as follows. Let t, m, n be positive integers with $t \le \min\{m, n\}$, let $X = \big(x_{ij}\big)$ be an $m \times t$ matrix of indeterminates over K, and let $Y = \big(y_{jk}\big)$ be a $t \times n$ matrix of indeterminates over K. Let $G = \mathrm{GL}(t, K)$ act on $K[X, Y]$ as follows: if $A \in G$, A acts by sending the entries of X to the entries of XA^{-1} and the entries of Y to the entries of AY. The preceding example is the case where $t = 1$. It is proved, for example, in Weyl's book that the ring of invariants is generated by the entries of the $m \times n$ product matrix XY. These entries are the scalar products of the various rows of X with the various columns of Y. It is clear that then entries of XY *are* invariant, because $(XA^{-1})(AY) = XY$. Again, one can map $K[U] \twoheadrightarrow K[XY] = R^G$ as K-algebras, where U is an $m \times n$ matrix of new indetermnates, and it is easy to show that the ideal generated by the $(t+1) \times (t+1)$ size minors of U is in the kernel. It turns out that, in fact, $R^G = K[XY] \cong K[U]/I_{t+1}(U)$. The Theorem above then implies that $K[U]/I_{t+1}(U)$ is Cohen-Macaulay in characteristic 0. (This is true in all characteristics: see [M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971) 1020–1058].)

Third example. Let X be an $n \times s$ matrix of indeterminates over the field K, where $1 \le n \le s$, and let $G = \mathrm{SL}(n, K)$ act on on $K[X]$ be sending the entries of X to the corresponding entries of AX. Note that if C denotes any column of X, the entries of C are sent to the corresponding entries of AC. It follows that if Y is any $n \times k$ submatrix of X (so that Y consists of a set of columns of X), then the entries of Y are sent to the corresponding entries of AY. Consequently, if Y is any $n \times n$ submatrix of X, then $\det(AY) = \det(A)\det(Y) = \det(Y)$, since the elements of $\mathrm{SL}(n, K)$ are precisely the $n \times n$ matrics with determinant 1. In this case $R^G = K[X/n]$, the ring generated over K by the $\begin{pmatrix} s \\ n \end{pmatrix}$ $n \times n$ minors of X, the so-called *maximal* minors of X. The relations on the minors are generated by certain standard quadratic relations called the *Plücker relations*.[1]

By the Theorem above, these rings $K[X]^G = K[X/n]$ are Cohen-Macaulay in characteristic 0. (This is also true in characteristic $p > 0$: see for example, [M. Hochster,

[1]These rings are well-known in algebraic geometry: the set of n-dimensional vector subspaces of K^s has the structure of a projective algebraic variety, which can be embedded in a projective space over K of dimension $\binom{s}{n} - 1$. The idea is that given a subspace V, one can choose an $s \times n$ matrix M whose rows are a basis for V: the $\binom{s}{n}$ minors of this matrix do not all vanish, and satisfy the Plücker relations. Therefore they give a point in the algebraic set G defined by the Plücker relations. G turns out to be irreducible. If one changes the matrix, the new matrix can be gotten from M by multiplying on the left by an invertible $n \times n$ matrix A: each of the $n \times n$ minors of AM is the product of $\det(A)$ with the corresponding minor of M, and so one gets the same point in projective space no matter which matrix whose rows are a basis for V is chosen. It can be shown that every point of G can be obtained in this way from a unique subspace of K^s of dimension n, so that this gives a bijective correspondence between the projective variety G and the set of n-dimensional vector subspaces of K^s. The projective variety V is called the *Grassmann variety* or *Grassmannian*.

Grassmannians and their Schubert subvarieties are arithmetically Cohen-Macaulay, J. of Algebra **25** (1973) 40–57]. They are also known to be unique factorization domains.

We shall also deduce from the Theorem stated above that an integrally closed ring that is a subring of $Kx_1, \ldots, x_n]$ generated by monomials is Cohen-Macaulay. In general, normality is far from sufficient for the Cohen-Macaulay property. The proof we give will depend on showing that any such ring is isomorphic with a ring of invariants of an algebraic torus, i.e., a product of copies of $\mathrm{GL}(1, K)$, acting on a polynomial ring. Cf. [M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Annals of Math. **96** (1972) 318–337].

Monomial submodules and the colon operation

Our next objective is to use revlex to give a criterion for when a sequence of indeterminates is a regular sequence on a module. We need some preliminaries concerning the behavior of monomial submodules and the colon operation.

If $M \subseteq F$ are any two R-modules and J is an ideal of R, we define

$$M :_F J = \{f \in M : Jf \subseteq M\}.$$

When $J = uR$ is the principal, we may write $M :_F u$ instead of $M :_F uR$.

When u is a nonzerodivisor (we shall typically be in this situation, for u will almost always be a nonzero element of a polynomial ring in the sequel), we have the following:

$$(*) \quad u(M :_F u) = M \cap uF \text{ and so } M :_F u = \frac{1}{u}(M \cap uF).$$

In fact, $uf \in M$ means precisely that $f \in M :_F u$, and then $f = \dfrac{1}{u}uf$ is uniquely determined.

We proved early that for monomial submodules and ideals, intersection distributes over sum. Hence $(*)$ yields:

Proposition. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K, and F a fintely generated free module. Let $M_1, \ldots, M_k$ be monomial submodules of F, and let $\mu \in R$ be a monomial. Then*

$$(M_1 + \cdots + M_k) :_F \mu = (M_1 :_F \mu) + \cdots + (M_k :_F \mu). \qquad \square$$

This gives a very easy way of calculating $M :_F \mu$ when M is a monomial module. If $\nu_j e_{i_j}$ is a typical generator, M is the sum of the modules $\nu_j e_{i_j} R$. It follows that $M :_F \mu$ is the sum of the modules $\nu_j e_{i_j} R :_F \mu$. Each of these is simply $(\nu_j R :_R \mu) e_{i_j}$. Thus, we have reduced to calculating $\nu R :_R \mu$ when ν and μ are monomials in R. Each of these is a cyclic module generated by one monomial, namely $\nu/\mathrm{GCD}(\mu, \nu)$.

An alternative description is as follows: if $a, b \in \mathbb{N}$, let $a \dot{-} b = \max\{a - b, 0\}$, and if $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n) \in \mathbb{N}^n$, let $\gamma = (a_1 \dot{-} b_1, \ldots, a_n \dot{-} b_n)$. Then $x^\alpha R : x^\beta = x^\gamma R$.

It is quite easy to see that a monomial μ is a nonzero divisor on F/M, where M is monomial, if and only if the variables occurring in μ do not occur in any minimal generator of M. This implies that $\mu_1, \ldots, \mu_h$ is a regular sequence on F/M if and only the variables occurring in μ_i do not occur in any other μ_j nor in any minimal generator of M.

We shall next aim to show that for reverse lexicographic order on F, if $M \subseteq F$ is graded, $x_{k+1}, \ldots, x_n$ is a regular sequence on F/M if and only if it is a regular sequence on $F/\mathrm{in}(M)$.