Math 615: Lecture of February 2, 2007

Regular sequences in the monomial case

We want to analyze what it means for a sequence of monomials $\mu_1, \ldots, \mu_k$ in $R = K[x_1, \ldots, x_n]$ to be a regular sequence on F/M when F is a finitely generated free R-module and M is a monomial submodule of F.

First note that, quite generally, $f \in R$ is not a zerodivisor on Q/N if and only if $N :_Q f = N$. This says precisely that $fu \in N$ if and only if $u \in N$. This yields:

Proposition. *Let $R = K[x_1, \ldots, x_n]$ and let $\mu_1, \ldots, \mu_k$ be a sequence of monomials in R. Let M be a monomial submodule of the finitely generated free module F. Then $\mu_1, \ldots, \mu_k$ is a regular sequence on F/M if and only if no variable that occurs in μ_i occurs in another μ_j, nor in any of the minimal monomial generators of M.*

Proof. Since $M = I_1e_1 \oplus \cdots I_se_s$ where the I_j are monomial ideals, we reduce at once to the case where $M = I$ is a monomial ideal: call the minimal monomial generators $\nu_1, \ldots, \nu_h$. We use induction on k. If $k = 1$, note that if μ_1 shares a variable x_t with ν_i then $\nu_i :_R \mu_1$ is generated by a monomial that divides ν_i and has a smaller exponent on x_t then ν_i does. This element is not in I, by the minimality of ν_i, but is in $I : \mu_1$. Hence the condition that μ_1 not involve a variable occurring in any ν_i is necessary. On the other hand, if that is true then $\nu_i :_R \mu = \nu_i R$ for every i, $1 \leq i \leq h$, and since colon distributes over sum we have that

$$I :_R \mu_1 = (\sum_{i=1}^{h} \nu_i R) :_R \mu_1 = \sum_{i=1}^{h} (\nu_i R :_R \mu_1) = \sum_{i=1}^{h} \nu_i R = I,$$

as required. Moreover it is clear that $\nu_1, \ldots, \nu_h, \mu_1$ are minimal generators for $I + \mu_1 R$. The inductive step is then an application of the case where $k = 1$. $\square$

Compatible orders and a sufficient condition for regularity of a sequence

Given a polynomial ring $K[x_1, \ldots, x_n]$ over a field K and a monomial order $>$ on a finitely generated R-free module F with ordered free basis $e_1, \ldots, e_s$, recall that for every t, $1 \leq t \leq s$, there is a monomial order $>_t$ on R defined by the condition $\mu > \mu'$ precisely if $\mu e_t > \mu' e_t$. Moreover, if $g \in R - \{0\}$ and $f \in F - \{0\}$ are such that $\mathrm{in}(f)$ involves e_t, then

$$(\dagger) \quad \mathrm{in}(gf) = \mathrm{in}_{>_t}(g)\mathrm{in}(f).$$

See the second page of the Lecture Notes of January 19. We shall say that a monomial order $>_R$ on R is *compatible* with a given monomial order $>$ on F if all of the orders $>_t$

are the same, and agree with $>_R$. It follows at once that if $>_R$ and $>$ on F are compatible, then for all $g \in R - \{0\}$ and $f \in F - \{0\}$,

$$(\dagger\dagger) \quad \mathrm{in}(fg) = \mathrm{in}_{>_R}(g)\mathrm{in}(f).$$

In fact, condition $(\dagger\dagger)$ is easily seen to be equivalent to compatibility. In working with compatible monomial orders, we typically use the same symbol $>$ for both.

If two of the $>_t$ are distinct, which can happen, there is no compatible order on R. If there is a compatible order on R, it is unique. The standard method of extending a monomial order on R to a monomial order on F (i.e., $\mu e_i > \mu' e_j$ if $\mu > \mu'$ or $\mu = \mu'$ and $i < j$) always produces a monomial order on F with which the original monomial order is compatible. In particular, revlex on F is compatible with revlex on R. In the sequel, when F is graded so that its generators do not necessarily all have degree 0, we give a slightly different way of extending revlex to F — but it is still compatible with revlex on R.

We next observe the following sufficient (but not necessary) condition for elements of R to be a regular sequence on F/M. Notice that we are not assuming that M is graded, nor that $>$ is revlex.

Theorem. *Let $R = K[x_1, \ldots, x_n]$, $f_1, \ldots, f_k \in R$ and let M be any submodule of a finitely generated free R-module F. Suppose that we have compatible monomial orders on R and F. If $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_k)$ form a regular sequence on $\mathrm{in}(M)$, then $f_1, \ldots, f_k$ is a regular sequence on M and, for $1 \le i \le k$, $\mathrm{in}\big(M+(f_1, \ldots, f_i)F\big) = \mathrm{in}(M)+\big(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_i)\big)F$.*

Proof. We use induction on k, and we consequently can reduce at once to the case where $k = 1$. We write f for f_1, and we must show that if $\mathrm{in}(f)$ is a not a zerodivisor on $F/\mathrm{in}(M)$ then (1) f is not a zerodivisor on F/M and (2) $\mathrm{in}(M + fM) = \mathrm{in}(M) + \mathrm{in}(f)F$.

If (1) fails we have $fu \in v \in M$ with $u \notin M$, and we can choose such an example with $\mathrm{in}(u)$ minimum, since the monomial order on F is a well-ordering. By the compatibility of orders, $\mathrm{in}(fu) = \mathrm{in}(f)\mathrm{in}(u) = \mathrm{in}(v) \in \mathrm{in}(M)$, and since $\mathrm{in}(f)$ is not a zerodivisor on $\mathrm{in}(M)$, we have that $\mathrm{in}(u) \in \mathrm{in}(M)$, so that we can choose $u' \in M$ with $\mathrm{in}(u) = \mathrm{in}(u')$. Then fu and fu' are both in M, and so $f(u - u') \in M$. But the initial terms of u and u' cancel, so that $u = u'$ or $\mathrm{in}(u - u') < \mathrm{in}(u)$. The latter contradicts the minimality of the choice of u, and the former shows that $u \in M$.

To prove (2), note that $\mathrm{in}(M) + \mathrm{in}(f)F \subseteq \mathrm{in}(M + fF)$ is obvious, and so we need only prove the opposite inclusion. If it fails, we can choose $u + fv \in M + fF$ where $u \in M$, $v \in F$, such that $\mathrm{in}(u + fv) \notin \mathrm{in}(M) + \mathrm{in}(f)F$, and, again, we can make this choice so that $\mathrm{in}(v)$ is minimum (note that v cannot be 0). We consider two cases.

First case: $\mathrm{in}(fv) \in \mathrm{in}(M)$. Then $\mathrm{in}(f)\mathrm{in}(v) \in \mathrm{in}(M)$ and, since $\mathrm{in}(f)$ is not a zerodivisor on $\mathrm{in}(M)$, we have that $\mathrm{in}(v) \in \mathrm{in}(M)$ and we can choose $v' \in M$ such that $\mathrm{in}(v) = \mathrm{in}(v')$. Then $u + fv = (u + fv') + f(v - v')$ still has initial form not in $M + fV$, and we have $u + fv' \in M$ while $v - v'$ has smaller initial form than v, a contradiction.

Second case: $\mathrm{in}(fv) \notin \mathrm{in}(M)$. In this case, $\mathrm{in}(fv)$ and $\mathrm{in}(u) \in \mathrm{in}(M)$ cannot cancel, and so one of them must be $\mathrm{in}(u + fv)$. But then either $\mathrm{in}(u + fv) = \mathrm{in}(u) \in \mathrm{in}(M)$ or $\mathrm{in}(u + fv) = \mathrm{in}(fv) = \mathrm{in}(f)\mathrm{in}(v) \in \mathrm{in}(f)F$, as required. $\square$

Special properties of reverse lexicographic order and a converse result

Throughout this section, $R = K[x_1, \dots, x_n]$ is a polynomial ring over K considered with reverse lexicographic order, F is a finitely generated graded free R-module with ordered free homogeneous basis $e_1, \dots, e_s$, also with reverse lexicographic order, which we define as follows. In the graded case we still want revlex to define total degree. Therfore, we define $\mu e_i >_{\mathrm{revlex}} \mu' e_j$ to mean either that (1) $\deg(\mu e_i) > \deg(\mu' e_j)$ or (2) $\deg(\mu e_i) = \deg(\mu' e_j)$ and $\mu < \mu'$ in lexicographic order for the variables ordered so that

$$x_n > x_{n-1} > \cdots > x_2 > x_1,$$

or (3) $\deg(\mu e_i) = \deg(\mu' e_j)$, $\mu = \mu'$, and $i < j$.

Let M be a graded submodule of F. We already noted at the end of the Lecture of January 31 that $x_{k+1}, \dots, x_n$ is a regular sequence on F/M if and only if $x_{k+1}, \dots, x_n$ is a regular sequence on $F/\mathrm{in}(M)$, which we know is equivalent to the condition that no minimal monomial generator of $\mathrm{in}(M)$ invovles any of the variables $x_{k+1}, \dots, x_n$. The preceding Theorem already shows that the condition is sufficient. We next want to prove that it is necessary as well. The following very easy result is a key fact about revlex that we shall use repeatedly.

Lemma. *Let notation be as above and let $u \in F - \{0\}$ be a homogeneous element. Then for every positive integer h, x_n^h divides u if and only if x_n^h divides* $\mathrm{in}(u)$.

Proof. "Only if" is obvious. The "if" part is immediate from the definition: since all terms have the same degree, any term not divisible by x_n^h is strictly larger than any term divisible by x_n^h. $\square$

Proposition. *Let notation be as above, with $M \subseteq F$ graded, and let $g_1, \dots, g_r$ be a Gröbner basis for M consisting of homogeneous elements. Let k be a positive integer.*

(a) $\mathrm{in}(M + x_n^h F) = \mathrm{in}(M) + x_n^h F$*, and $g_1, \dots, g_r, x_n^k e_1, \dots, x_n^k e_s$ is a Gröbner basis for $M + x_n^h F$.*

(b) $\mathrm{in}(M :_F x_n^h) = \mathrm{in}(M) :_F x_n^h$*. Moreover, if for $1 \le j \le r$, t_j denotes the greatest integer in the interval $[0, h]$ such that $x^{t_j} | g_j$ and $h_j = g_j / x_n^{t_j}$, then $h_1, \dots, h_r$ is a Gröbner basis for $M :_F x_n^h$.*

Proof. (a) Clearly, $\mathrm{in}(M) + x_n^h F \subseteq \mathrm{in}(M + x_n^h)F$. Now consider $\mathrm{in}(u + x_n^h f)$ where $u \in U$ and $f \in F$. In revlex, the homogeneous component of an element of highest degree has the same initial form as the element, and so we may assume that $u + x_n^h f$ is homogeneous. If the initial term is divisible by x_n^h the result is proved. If not, it must be a term of u, and x_n must occur with a strictly smaller exponent than h. All other terms of u must be smaller: either they are not divisible by x_n^h and persist in $u + x_n^h f$, or they are divisible by x_n^h, which forces them to be smaller than u in revlex, by the definitiion of revlex. The statement about the Gröbner basis is immediate, since the specified elements are in $M + x_n^h F$ and their initial terms span $\mathrm{in}(M) + x_n^h F$.

(b) We have that a monomial $\nu \in \mathrm{in}(M :_F x_n^h)$ iff and $x^h \nu \in \mathrm{in}(M)$ iff $x^h v = \mathrm{in}(w)$ with $w \in M$ homogeneous. But x_n^h divides w if and only x_n^h divides $\mathrm{in}(w)$, by the Lemma above, and the result is immediate. We then have that $\mathrm{in}(M)$ is the span of the $\mathrm{in}(g_j)R : Fx_n^h$, and these are the same as the $\mathrm{in}(g_j/x_j^{t_j})R$. Again, we are using that a power of x_n divides g_j if and only if it divides $\mathrm{in}(g_j)$. $\square$

We can now prove:

Theorem. *Let notation be as above, with $M \subseteq F$ graded, and use revlex order on F and R. Then $x_{k+1}, \dots, x_n$ is a regular sequence on F/M if and only if it is a regular sequence on $F/\mathrm{in}(M)$.*

Proof. Since regular sequences are permutable in the graded case, we may show instead the same result for $x_n, \dots, x_{k+1}$. We already know the "if" part. Now suppose that x_n is not a zerodivisor on F/M. Then $M :_F x_n = M$, and so

$$\mathrm{in}(M) = \mathrm{in}(M :_F x_n) = \mathrm{in}(M) : Fx_n = \mathrm{in}(M).$$

The proof is now completed by induction: when we work mod x_n, R is replaced by $R/x_nR = K[x_1, \dots, x_{n-1}]$, F by F/x_nF, and M by $M/x_nM \hookrightarrow F/x_nF$, since x_n is not a zerodivisor on M/x_nM. The hypothesis is preserved because of the preceding Proposition. $\square$