**Math 615: Lecture of February 5, 2007**

**Associated primes and primary decompostion for modules**

Throughout this section $R$ is a Noetherian ring and $M$ an $R$-module. Recall that $P$ is an *associated prime* of $M$ if, equivalently

(1) There is an injection $R/P \hookrightarrow M$.

(2) There is an element $u \in M$ such that $\mathrm{Ann}_R u = P$.

The set of associated primes of $M$ is denoted $\mathrm{Ass}\,(M)$. Although we have made this definition even when $M$ need not be finitely generated, the rest of our study is restricted to the case where $M$ is Noetherian. Note that if $M = 0$, then $\mathrm{Ass}\,(M) = \emptyset$. The converse is also true, as we shall see below.

**Proposition.** *Let $M$ be a finitely generated $R$-module, where $R$ is Noetherian*

(a) *If $u \neq 0$ is any element of $M$, one can choose $s \in R$ such that $\mathrm{Ann}_R su$ is a prime ideal $P$ of $R$, and $P \in \mathrm{Ass}\,(M)$. In particular, if $M \neq 0$, then $\mathrm{Ass}\,(M)$ is nonempty.*

(b) *If $ru = 0$ where $r \in R$ and $u \in M - \{0\}$, then one can choose $s \in R$ such that $\mathrm{Ann}_R su = P$ is prime. Note that $r \in P$. Consequently, the set of elements of $R$ that are zerodivisors on $M$ is the union of the set of associated primes of $M$.*

(c) *If $M \neq 0$ it has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ in which all the factors $M_i/M_{i-1}$ for $1 \leq i \leq n$ are prime cyclic modules, i.e., have the form $R/P_i$ for some prime ideal $P_i$ of $R$.*

(d) *If $N \subseteq M$, then $\mathrm{Ass}\,(N) \subseteq \mathrm{Ass}\,(M)$.*

(e) *If $0 \to M' \to M \to M'' \to 0$ is exact, then $\mathrm{Ass}\,(M) \subseteq \mathrm{Ass}\,(M') \cup \mathrm{Ass}\,(M'')$.*

(f) *If $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ is a finite filtration of $M$, then*

$$\mathrm{Ass}\,(M) \subseteq \bigcup_{i=1}^{n} \mathrm{Ass}\,(M_i/M_{i-1}).$$

(g) *If one has a prime cyclic filtration of $M$ as in part (c), $\mathrm{Ass}\,(M) \subseteq \{P_1, \ldots, P_n\}$. In particular, $\mathrm{Ass}\,(M)$ is finite.*

(h) *If $W$ is a multiplicative system in $R$, $\mathrm{Ass}\,(W^{-1}M)$ over $W^{-1}M$ is the set*

$$\{PW^{-1}R : P \in \mathrm{Ass}\,(M) \text{ and } P \cap W = \emptyset\}.$$

*Proof.* (a) The family of ideals $\{\mathrm{Ann}_R tu : t \in R \text{ and } tu \neq 0\}$ is nonempty since we may take $t = 1$. Since $R$ has ACC, it has a maximal element $\mathrm{Ann}_R su = P$. We claim that $P$

is prime. If $ab \in P$, then $abu = 0$. If $a \notin P$, we must have $b \in P$, or else $bu \neq 0$ and has annihilator containing $P + aR$ strictly larger than $P$.

(b) This is immediate from (a). Note that it is obvious that if $P = \mathrm{Ann}_R u$ with $u \in M$, then $u \neq 0$, and so every element of $P$ is a zerodivisor on $M$.

(c) Choose a sequence of elements $u_1, u_2, \cdots$ in $M$ recursively as follows. Choose $u_1$ to be any element of $M$ such that $\mathrm{Ann}_R u_1 = P_1$ is prime. If $u_1, \ldots, u_i$ have been chosen and $Ru_1 + \cdots + Ru_i = M$, the seqeunce stops. If not, choose $u_{i+1} \in M$ such that its image $\overline{u}_{i+1}$ in $M/(Ru_1 + \cdots + Ru_i)$ has annihilator $P_{i+1}$ that is prime. Let $M_i = Ru_1 + \cdots + Ru_i$. The sequence must stop, since the $M_i$ are strictly increasing and $M$ has ACC. By construction, the factors are prime cyclic modules.

(d) This is obvious, since if $R/P \hookrightarrow N$, we have a composite map $R/P \hookrightarrow N \hookrightarrow M$.

(e) Let $u \in M$ be such that $\mathrm{Ann}_R u = P$, which means that $Ru \cong R/P$. If $Ru$ meets $M' - \{0\}$, say $ru = v$ is a nonzero element of $M'$, then $\mathrm{Ann}_R v = P$ since $v$ may be thought of as a nonzero element of $R/P$, and $P \in \mathrm{Ass}\,(M')$. If $Ru \cap M' = 0$, then the composite map $R/P \cong Ru \subseteq M \twoheadrightarrow M''$ is injective, and so $P \in \mathrm{Ass}\,(M'')$.

(f) We use induction on $n$. By part (e),

$$\mathrm{Ass}\,(M) \subseteq \mathrm{Ass}\,(M_{n-1}) \cup \mathrm{Ass}\,(M/M_{n-1})$$

and we may apply the induction hypothesis to $0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1}$.

(g) This is immediate from part (f), since $\mathrm{Ass}\,(R/P_i) = \{P_i\}$.

(h) If $R/P \hookrightarrow M$ and $P$ does not meet $W$, then $W^{-1}R/PW^{-1}R \hookrightarrow W^{-1}M$. Conversely, suppose that $u/w_0 \in W^{-1}M$ where $u \in M$ and $w_0 \in W$ has annihilator $Q$ in $W^{-1}R$. The same is true for $w_0(u/w_0) = u/1$. We know that $Q = PW^{-1}R$ for some prime $P$ of $R$ such that $P \cap W = \emptyset$. Choose $w \in W$ such that $\mathrm{Ann}_R wu$ is maximal. If $f \in P$, we know that $fwu/1$ is 0 in $W^{-1}M$, and so we can choose $v \in W$ such that $vfwu = 0$. But $\mathrm{Ann}_R(vwu) = \mathrm{Ann}_R(wu)$ by the maximality of $\mathrm{Ann}_R(wu)$, so that we must have $fwu = 0$. On the other hand, if $fwu = 0$ for $f \in R$, then $f/1 \in Q$, and so $f$ is in the contraction of $Q$ to $R$, which is $P$. We have shown that $P = \mathrm{Ann}_R(wu)$, and so $P \in \mathrm{Ass}\,(M)$. $\square$

*Remark.* If $M$ is nonzero module over a Noetherian domain $R$, then $M$ is torsion-free over $R$ if and only if $\mathrm{Ass}\,(M) = \{(0)\}$, since this says precisely that no nonzero element of $R$ is a zerodivisor on $M$.

*Remark.* There does not necessarily exist a filtration of $M$ with prime cyclic factors in which the only primes that occur are associated primes of $M$. For example, let $R = K[x, y]$ be the polynomial ring in two variables over a field $K$ and let $M = (x, y)R \subseteq R$, which is an ideal of $R$, but which we are viewing as a torsion-free $R$-module. Then $\mathrm{Ass}\,(M) = (0)$, but there is no finite filtration of $M$ in which every factor is $R$, since $M$ needs two generators but is rank one, and so is not free over $R$.

Recall that if $M$ is finitely generated over a Noetherian ring $R$ and $I = \mathrm{Ann}_R M$, then $P \in \mathrm{Supp}\,(M)$, which means that $M_P \neq 0$, if and only if $I \subseteq P$. The minimal primes of

Supp $(M)$ are the same as the minimal primes of $I$, and are called the *minimal primes* of $M$.

**Proposition.** *Let $M$ be a finitely generated module over a Noetherian ring $R$, and let $I = \mathrm{Ann}_R M$.*

(a) *Every associated prime of $R/I$ is an associated prime of $M$.*

(b) *Every associated prime of $M$ contains a minimal prime of $M$. Every minimal prime of $M$ is an associated prime of $M$, and so the minimal primes of $M$ are the same as the minimal primes of* Ass $(M)$.

(c) *Let $m$ be a maximal ideal of $R$. Then the following conditions on $M$ are equivalent:*

(1) Ass $(M) = \{m\}$.

(2) Supp $(M) = \{m\}$.

(3) *$M$ is killed by a power of $m$.*

(4) *$M$ has a finite filtration in which all the factors are $\cong R/m$.*

*Proof.* (a) Let $u_1, \ldots, u_h$ generate $M$. The map $R \to M^h$ that sends $r \mapsto (ru_1, \ldots, ru_h)$ has kernel $I$, yielding an injection $R/I \hookrightarrow M^h$. Since $M \subseteq M^h$, Ass $(M) \subseteq$ Ass $(M^h)$. Since $M_h$ has a finite filtration $0 \subseteq M \subseteq M \oplus M \subseteq \cdots \subseteq M^{h-1} \subseteq M^h$ in which all factors are $M$, Ass $(M^h) \subseteq$ Ass $(M)$. Thus, Ass $(R/I) \subseteq$ Ass $(M^h) =$ Ass $(M)$.

(b) Since $R/P \hookrightarrow M$, we have that $I$ kills $R/P$, and so $I \subseteq P$, so that $P$ contains a minimal prime of $I$. Every minimal prime of $M$ is a minimal prime of $R/I$ and, hence, an associated prime of $R/I$. Therefore every minimal prime of $M$ is an associated prime of $M$ by part (a). The final statement is now clear.

(c) (1) $\Leftrightarrow$ (2) since in both cases $m$ is the only minimal prime of $M$. This implies that Rad $(I) = m$, and so $m^h \subseteq I$ for some $h$ and (2) $\Rightarrow$ (3). If $m^h M = 0$, $M$ has a finite filtration $0 = m^h M \subseteq m^{h-1} M \subseteq \cdots \subseteq m^2 M \subseteq mM \subseteq M$ and each factor $m^i M/m^{i-1} M$ is killed by $m$, and so is a finite-dimensional vector space over $K = R/m$. Hence, this filtration can be refined to one in which every factor is $\cong R/m$, since every $m^i M/m^{i-1} M$ has a finite filtration in which all factors are $\cong R/m$. Thus (3) $\Rightarrow$ (4). Finally, (4) $\Rightarrow$ (1) by part (g) of the earlier Proposition. □

If $P$ is a prime ideal of $R$, $M$ is called *$P$-coprimary* if, equivalently,

(1) Ass $(M) = \{P\}$.

(2) $M \neq 0$, for some $h \geq 1$, $P^h M = 0$, and every element of $R - P$ is a nonzerodivisor on $M$.

(3) $M \hookrightarrow M_P$ is injective, and $M_P$ has finite length over $R_P$.

We need to check that these three conditions are equivalent. (1) $\Rightarrow$ (3), for if Ass $(M) = P$ all elements of $R - P$ are nonzerodivisors on $M$ and $M \hookrightarrow M_P$. But since Ass$\,(M_P) = \{PR_P\}$ by part (h) of the Proposition on p. 1, and $PR_P$ is maximal in $R_P$, this implies that $M_P$ has finite length by the equivalence of (1) and (4) in part (c) of the preceding Proposition.

Assume (3). Then $(PR_P)^h$ kills $M_P$ for some $h$, and so $P^h$ kills $M \hookrightarrow M_P$. Since the elements of $R-P$ act invertibly on $M_P$, they are not zerodvisors on $M \subseteq R_P$. This shows that (3) $\Rightarrow$ (2).

Finally, assume (2). Choose $k$ as large as possible such that $P^kM \neq 0$: we allow $k = 0$. By hypothesis, $k \leq h-1$. Choose $u \neq 0$ in $P^kM$. Then $Pu \subseteq P^{k+1}M = 0$, while no element of $R-P$ kills $u$. It follows that $P \in$ Ass$\,(M)$. Moreover $P^h \subseteq \mathrm{Ann}(M)$ shows that every associated prime contains of $M$ contains $P$. But there cannot be an associated prime strictly larger than $P$, since it would contain an element of $R-P$, and such an element is a nonzerodivisor on $R$. Hence, (2) $\Rightarrow$ (1), as required. $\square$

*Remark.* When $M = R/I$ with $R$ a proper ideal of $R$, it is easy to see that $M$ is $P$-coprimary if and only if $I$ is primary to $P$.

We shall say that a proper submodule $N$ of $M$ is *irreducible* if it is not the intersection of two strictly larger submodules of $M$. It is easy to see that this is equivalent to the condition that $N$ not be the intersection of finitely many larger submodules of $M$. Note that in each part of the Lemma below, we can replace $M$ by $M/N$, $N$ by 0, and each submodule of $M$ containing $N$ by its image modulo $N$ without affecting any relevant issue.

**Lemma.** *Let $R$ be a Noetherian ring and let $N \subset M$ be finitely generated $R$-modules, where the inclusion is strict.*

(a) *$N$ is a finite intersection of irreducible submodules of $M$ (this includes the possibility that $N$ itself is irreducible).*

(b) *If $N$ is irreducible, then $M/N$ it is $P$-coprimary for some prime $P$.*

(c) *If $N_1, \ldots, N_k$ are submodules such that each $M/N_j$ is $P$-coprimary to $P$ for the same prime $P$, then $M/\bigcap_{j=1}^k N_j$ is also $P$-coprimary.*

*Proof.* (a) Let $\mathcal{N}$ denote the set of proper submodules of $M$ that are not finite intersections of irreducible submodules. If $\mathcal{N}$ is nonempty, it has a maximal element $N$. Then $N$ cannot itself be irreducible. Suppose that $N = N_1 \cap N_2$ where $N_1$ and $N_2$ are strictly larger. Then each $N_i$ is a finite intersection of strictly larger submodules, and, hence, so is $N_1 \cap N_2 = N$, a contradiction.

(b) We replace $M$ by $M/N$ and so assume that $N = 0$. If Ass$\,(M)$ contains two or more relevant primes, then we can choose $u \in M$ such that $\mathrm{Ann}_R u = P$ and $v \in M$ such that $\mathrm{Ann}_R v = Q$, where $P \neq Q$ are distinct primes. Then $Ru \cap Rv$ must be 0: any nonzero element of $Rv$ has annihilator $P$, while any nonzero element of $Ru$ has annihilator $Q$. This contradicts the irreducibility of 0.

(c) The map $M \to \prod_{j=1}^k M/N_j$ that sends $u \mapsto (u+N_1, \cdots, u+N_k)$ has kernel $N = \bigcap_{j=1}^k N_j$, and so we have $M/(\bigcap_{j=1}^k N_j) \hookrightarrow \prod_{j=1}^k N_j \cong \bigoplus_{j=1}^k (M/N_j)$ The latter has a filtration by submodules $M/N_1 \oplus \cdots \oplus M/N_j$ with factors $M/N_1,\, M/N_2,\, \ldots,\, M/N_k$. Hence, $\mathrm{Ass}\,(N) \subseteq \bigcup_{j=1}^k \mathrm{Ass}\,(M/N_j) = \{P\}$, as required. $\square$

If $N \subset M$ is a strict inclusion of finitely generated modules over a Noetherian ring $R$, we shall say that $N = N_1 \cap \cdots \cap N_k$ is a *primary decomposition* for $N$ in $M$ if

(1) Each $M/N_i$ is $P_i$-coprimary for some prime $P_i$ of $R$.

(2) If $i \neq j$ then $P_i \neq P_j$.

(3) The intersection is *irredundant* in the sense that if any $N_j$ is omitted, the intersection of the others is strictly larger than $N$.

**Theorem (primary decomposition for modules).** *Let $R$ be a Noetherian ring and let $N \subset M$, where the inclusion is strict. Then $N$ has a primary decomposition. In any primary decomposition, the primes occurring are precisely the elements of* $\mathrm{Ass}\,(M/N)$*, and the number of terms is the number of primes in* $\mathrm{Ass}\,(M/N)$*. If $P_i$ is a minimal prime of $M/N$, then the corresponding $P_i$-coprimary module $N_i$ in the primary decomposition is uniquely determined and is, in fact, $Ker\big(M \to (M/N)_{P_i}\big)$.*

*Proof.* To prove existence, first write $N$ as a finite intersection of irreducibles $N_j$, by part (a) of the preceding Lemma. For each prime $P$ such that one of these is coprimary to $P$, replace those $N_j$ that are $P$-coprimary by their intersection. Thus, $N$ is an intersection of $P$-coprimary modules such that the primes that occur are mutually distinct. If the intersection is not irredundant, we may successively omit terms until we reach an intersection that is irredundant.

We now want to prove the uniqueness statement. We pass to $M/N$ and so assume that $N = 0$. Suppose that $0 = N_1 \cap \cdots \cap N_k$ is a primary decomposition for 0 in $M$, where $M/N_j$ is $P_j$-coprimary. As in part the proof of part (c) of the preceding Lemma, we have an injection $M \hookrightarrow \bigoplus_{j=1}^k (M/N_j)$, and it folows that $\mathrm{Ass}\,(M)$ is contained in the set of primes $\{P_1,\, \ldots,\, P_k\}$. To see that $P_i \in \mathrm{Ass}\,(M)$, note that since the intersection is irredundant, we can choose an element $u \in \bigcap_{j \neq i} N_j - N_i$. The image of $u$ under $M \hookrightarrow \bigoplus_{j=1} M/N_j$ is 0 in every $M/N_j$ except $M/N_i$, and is nonzero in $M/N_i$. Hence,

$$\mathrm{Ass}\,(Ru) \subseteq \mathrm{Ass}\,(M/N_i) = \{P_i\},$$

and so $\mathrm{Ass}\,(Ru) = \{P_i\}$. But $Ru \subseteq M$, and so $\mathrm{Ass}\,(Ru) \subseteq \mathrm{Ass}\,(M)$, i.e., $P_i \in \mathrm{Ass}\,(M)$. The statement about the number of terms is now obvious.

Finally, suppose that $P = P_i$ is minimal among the associated primes. For every $P_j \neq P_i$, $P_j - P_i$ is nonempty. It follows that $(M/N_j)_{P_i} = 0$, so that $M_{P_i} = (N_j)_{P_i}$. Now,

$$N_{P_i} = (N_1 \cap \cdots \cap N_k)_{P_i} = (N_1)_{P_i} \cap \cdots \cap (N_k)_{P_i} = (N_i)_{P_i},$$

so that $(M/N)_{P_i} \cong (M/N_i)_{P_i}$. Hence, the kernel of $M \to (M/N)_{P_i}$ is the set of $u \in M$ such that for some $w \in R - P_i$, $wu \in N_i$. Since $M/N_i$ is $P_i$-coprimary, no element of $R - P_i$ is a zerodivisor on $M/N_i$, it follows that the kernel is $N_i$. $\square$

## Depth

We give a brief introduction to the theory of depth without using homological methods: the homological proofs of certain results, such as the fact that maximal regular sequences in $I$ on a module $M$ all have the same length, are very slick, but in some ways mask the simplicity of what is going on.

We shall assume that $R \to S$ is a homomorphism of Noetherian rings, that $I$ is an ideal of $R$, and that $M$ is a finitely generated $S$-module. By far the most important case is the one where $R = S$, and the reader is encouraged to focus on this situation if this is a first encounter with depth. The greater generality is very useful, however, in that one can frequently choose regular sequences that arise from a "smaller" ring.

If $IM = M$, we define the *depth* of $M$ on $I$ to be $+\infty$. If $IM \neq M$, it turns out that all maximal regular sequences on $M$ consisting of elements of $I$ have the same length, and we define this length to be the *depth* of $M$ on $I$. This fact is proved below. Before giving the proof, we want to characterize the "degenerate" situation in which $IM = M$ in a down-to-earth way.

**Proposition.** *Let $R \to S$ be a homomorphism of Noetherian rings, let $N$ be a finitely generated $R$-module, and let $M$ a finitely generated $S$-module. Let $I$ be the annihilator of $N$ in $R$, and let $J$ be the annihilator of $M$ in $S$.*

(a) *The support of $N \otimes_R M$ over $S$ is $\mathcal{V}(IS + J) = \{Q \in \mathrm{Spec}\,(S) : IS + J \subseteq Q\}$. In particular, $N \otimes_R M = 0$ if and only if $IS + J = S$, the unit ideal.*

(b) *In particular, if $N = R/I$ is cyclic, $IM = M$ if and only if $IS + J = S$.*

*Proof.* (a) Since $I$ kills $N$, $IS$ kills $N \otimes_R M$, and since $J$ kills $M$, $J$ kills $N \otimes_R M$. Thus, any prime in the support of $N \otimes_R M$ must contain $IS + J$. Now suppose that $IS + J \subseteq Q$, a prime of $S$, and the $Q$ lies over $P$ in $R$. It suffices to see that $(N \otimes_R M)_Q \neq 0$, and this may be identified with $N_P \otimes_{R_P} M_Q$. Here, $I \subseteq P$, and so $N_P \neq 0$. Let $R_P/PR_P = K$. By Nakayama's Lemma, $N_P/PN_P$ is a nonzero $K$-vector space, say $K^h$, $h \geq 1$, and, similarly, $M_Q/QM_Q$ is a nonzero vector space over $L = S_Q/QS_Q$, say $L^k$. Then $N_P \otimes_{R_P} M_Q$ maps onto $K^h \otimes_K L^k \cong (K \otimes_K L)^{hk} \cong L^{hk} \neq 0$, as required.

(b) This is immediate from part (a), since $N \otimes_R M \cong M/IM$ in this case. $\square$

We will need the following:

**Lemma.** *Let $R$ be a ring and let $x_1, \ldots, x_n$ be a regular sequence on an $R$-module $M$. Suppose that $x_2$ is not a zerodivisor on $M$. Then $x_2, x_1, x_3, x_4, \ldots, x_{n-1}, x_n$ is a regular sequence on $M$.*

*Proof.* It suffices to show that $x_1$ is a nonzerodivisor modulo $x_2M$: since $M/(\boldsymbol{x}_1, x_2)M = M/(x_2, x_1)M$, the remaining conditions are unaffected by the interchange of $x_2$ and $x_1$. Suppose that $x_1u \in x_2M$, say $x_1u = x_2v$. Since $x_1, x_2$ is a regular sequence on $M$ and $x_2v \equiv 0 \bmod x_1M$, we have that $v \in x_1M$, say $v = x_1w$. Then $x_1u = x_2x_1w$, and $x_1(u - x_2w) = 0$. Since $x_1$ is not a zerodivisor on $M$, $u \in x_2M$, as required. $\square$

We can now justify the definition we want to give for depth.

**Theorem.** *Let $R \to S$ be a homomorphism of Noetherian rings, let $M$ be a finitely generated $S$-module, and let $I \subseteq R$ be an ideal of $R$. Assume that $IM \neq M$.*

(a) *There is no infinite regular sequence $x_1, x_2, x_3, \ldots$ on $M$ consisting of elements of $I$.*

(b) *There is no zerodivisor on $M$ in $I$ if and only if $I$ is contained in the contraction of a prime in $\mathrm{Ass}\,(M)$ to $R$. Hence, there is a no nonzerodivisor on $M$ in $I$ if and only if there is an element $u \in M - \{0\}$ such that $Iu = 0$.*

(c) *Every regular sequence in $I$ on $M$ (including the empty regular sequence) can be extended to a maximal regular sequence in $I$ on $M$, and this maximal regular sequence is always finite.*

(d) *All maximal reqular sequences in $I$ on $M$ have the same length.*

*Proof.* (a) Suppose we have such a sequence. Let $I_n = (x_1, \ldots, x_n)R$. Since $R$ is Noetherian, we eventually have $I_n = I_{n+1}$. This means that $x_{n+1} \in I_n$, and so kills $M/I_nM$. Since $x_{n+1}$ is not a zerodivisor on $M/I_nM$, we must have $M/I_nM = 0$, i.e., $M = I_nM$. But $I_n \subseteq I$ and $M \neq IM$, a contradiction.

(b) Let $\theta$ denote the map $R \to S$. Note that the action of $x \in R$ on $M$ is the same as the action of $\theta(x)$. Hence, $x \in R$ is a zerodivisor on $M$ if and only if $\theta(x)$ is a zerodivisor on $M$, and this means that $\theta(x)$ is in the union of the associated primes $Q_1, \ldots, Q_k$ of $M$ in $S$. Let $P_i$ denote the contraction of $Q_i$ to $R$. We then have that $I$ consists entirely of zerodivisors on $M$ if and only if it is contained in the union of the $P_i$. But then it is contained in some $P_i$. Choose $u \in M - \{0\}$ such that $\mathrm{Ann}_S u = Q_i$. Then, since $\theta(I) \subseteq Q_i$, $Iu = 0$, as required.

(c) Suppose that we have a regular sequence $x_1, \ldots, x_k$ and that $I_k$ is the ideal $(x_1, \ldots, x_k)R$. If every element of $I$ is a zerodivisor on $M/I_kM$, then we have constructed the required maximal regular sequence on $M$ in $I$. If not, we can enlarge the regular sequence to $x_1, \ldots, x_{k+1}$ by taking $x_{k+1}$ to be an element of $I$ that is not a zerodivisor on $M/I_kM$. We can continue recursively in this way. The process must terminate by part (a).

(d) Suppose that we have a counterexample. Since $M$ has ACC, among all submodules $N$ of $M$ such that $M/N$ provides a counterexample, there is a maximal one. (The family is nonempty, since it contains 0.) Therefore, we may assume the result holds for every proper homomorphic image of $M$. If $I$ consists entirely of zerodivisors on $M$, the empty sequence is the unique maximal regular sequence on $M$.

Now suppose that $x \in I$ is a maximal regular sequence on $M$. Then $I$ consists entirely of zerodivisors on $M/xM$, and by part (b), there exists an element $u \in M - xM$ such that $Iu \subseteq xM$. Now let $y \in I$ be a nonzerodivisor. We want to show that it constitutes a maximal regular sequence. Since $Iu \subseteq xM$, we can write $yu = xv$ for $v \in M$. First note that $v \notin yM$, for if $v = yw$, then $yu = xyw$. Since $y$ is a nonzerodivisor, this implies, $u = xw$, a contradiction. The argument in this case will therefore be complete if we can show that $Iv \subseteq yM$. But if $f \in I$, we have $xfv = f(xv) = f(yu) = y(fu) = y(xw)$ for some $w \in M$, since $Iu \subseteq xM$. But then $x(fv - yw) = 0$, and since $x$ is not a zerodivisor on $M$, we have that $fv = yw \in yM$, as required.

Finally, suppose that we have two maximal regular sequences $x_1, \ldots, x_h$ and $y_1, \ldots, y_k$ on $M$ in $I$ where $h \geq 2$ and $k \geq 2$. Then the contractions to $R$ of the associated primes of $M/x_1M$ do not cover $I$ (they miss $x_2$), and the contractions to $R$ of the associated primes of $M/y_1M$ do not cover $I$ similarly. Likewise, the contractions of the associated primes of $M$ do not cover $I$ (they miss $x_1$). It follows that the union of all three sets of primes does not cover $I$: if it did, $I$ would be contained in one of these primes, a contradiction. We can therefore pick $z \in I$ not in any of them. Then $x_1$, $z$ is a regular sequence on $M$, and can be extended to a maximal regular sequence on $M$ in $I$, say $x_1, z, x'_3, \ldots, x'_{h'}$. Similarly, we can construct a maximal sequence on $M$ in $I$ of the form $y_1, z, y'_3, \ldots, y'_{k'}$.

But if two maximal regular sequence on $I$ in $M$ have the same first term, say $x$, then the terms after the first form maximal regular sequences on $M/xM$, a proper quotient of $M$. It follows that they have the same length, since we know the result for $M/xM$, and so the original regular sequences have the same length. Thus, $h' = h$ and $k' = k$. By the Lemma above, $z, x_1, x'_3, \ldots, x'_h$ is also a regular sequence on $M$, and so is $z, y_1, y'_3, \ldots, y'_k$. Since these two have the same first term, we obtain that $h = k$. $\square$

We are now justified, under the hypotheses of the Theorem above, in defining the *depth* of $M$ on $I$, which we shall denote $\mathrm{depth}_I M$, to be the length of any maximal regular sequence on $M$ whose terms are in $I$.

We also note:

**Proposition.** *Let $R$ be a finitely generated $\mathbb{N}$-graded algebra with $R_0 = K$, a field, let $m = \bigoplus_{d=1}^{\infty} R_d$ be the homogeneous maximal ideal, and let $M$ be a finitely generated $\mathbb{Z}$-graded $R$-module. Then the depth of $M$ on $m$ is the same as the length of any maximal regular sequence on $M$ consisting of forms of positive degree. Hence, $R$ is Cohen-Macaulay if and only if* $\mathrm{depth}_m R = \dim(R)$.

*Proof.* First note that if $\mathrm{depth}_m M > 0$, then we can construct a nonzero form $F_1$ of positive degree that is not a zerodivisor on $M$, by homogeneous prime avoidance. We can then proceed recursively to construct a maximal regular sequence of such forms on $M$: we begin by passing to $M/F_1M$. The final statement is now obvious. $\square$