Math 615: Lecture of February 12, 2007

We next want to consider one example where the generic initial ideal depends on the characteristic. The example also illustrates that, even when the given ideal is monomial, the generic initial ideal can be rather different.

Consider $I = (x_1^2, x_2^2)$ in $R = K[x_1, x_2]$ where K is infinite. Suppose that we use either hlex or revlex as the monomial order. If $A = \bigl(a_{ij}\bigr)$,

$$\mathrm{Gin}(I) = \mathrm{in}\bigl(((a_{11}x_1 + a_{21}x_2)^2, (a_{12}x_1 + a_{22}x_2)^2)R\bigr)$$

for a_{11}, a_{12}, a_{21}, a_{22} in sufficiently general position. In characteristic different from 2, we get x_{11}^2 as the initial term from either generator. The initial term of

$$a_{12}^2(a_{11}x_1 + a_{21}x_2)^2 - a_{11}^2(a_{12}x_1 + a_{22}x_2)^2$$

yields an x_1x_2 term. In degree $d \geq 3$, I contains all monomials of degree d, and, hence, so does AI. It follows that $\mathrm{Gin}(I) = (x_1^2,\, x_1x_2, x_2^3)$. However, in characteristic two, both squares are linear combinations of x_1^2 and x_2^2, and $\mathrm{Gin}(I) = (x_1^2, x_2^2)$. This is consistent with out characterization of Borel-fixed ideals because it is false that $1 \leq_2 2$: the binomial coefficient $\binom{2}{1}$ vanishes modulo 2.

The following result explains in part why generic initial ideals have great interest.

Theorem. *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let I be a homogeneous ideal of R. Let m be the homogeneous maximal ideal of R. Then* $\mathrm{depth}_m(R/I) = k$ *if and only if the minimal monomial generators of* $\mathrm{Gin}(R/I)$ *for revlex involve x_{n-k} but not x_j for $j \geq n-k+1$.*

Proof. After the variables are placed in general position, say by a change of coordinates using a matrix of indeterminates, there is a regular sequence of length k on R/I if and only if the last k variables form such a sequence: see the Proposition on p. 3 of the Lecture Notes of February 7. By our results on reverse lexicographic order, this is equivalent to the absence of the last k variables from the initial ideal with respect to revlex: see the final Theorem of the Lecture of February 2. $\square$

Actions on vector spaces and exterior algebra

We are aiming to prove results concerning when the initial ideal is Borel-fixed. The theorems we obtain can actually be viewed as results about actions on on finite-dimensional K-vector subspaces of R.

We assume that $R = K[x_1, \ldots, x_n]$, a polynomial ring over an infinite field K, and we fix a monomial order on R such that $x_1 > x_2 > \cdots > x_n$, as usual.

This gives an ordered basis for every subspace of R spanned by monomials. Recall that when a vector space V has an ordered basis $v_1 > v_2 \cdots > v_h$, the theory of Gröbner bases applies directly to V: the base ring may be thought of as K, the polynomials in 0 variables over the field K. When V is a subspace of R, this gives us, *a priori*, two notions of initial term. We write $\mathrm{in}_{\mathrm{vec}}(f)$ to indicate that we are taking the initial term in a vector space sense. However, in practice, we shall frequently be considering a finite-dimensional K-vector subspace of R spanned by monomials, with the order of the basis elements obtained by restricting the monomial order on R. In this case, $\mathrm{in}(f)$ and $\mathrm{in}_{\mathrm{vec}}(f)$ agree. However, $\mathrm{in}_{\mathrm{vec}}(W)$ is a K-vector subspace of V, not an ideal of R.

Recall that the exterior algebra $\bigwedge^{\bullet}(V) = \bigoplus_{k \in \mathbb{N}} \bigwedge^k(V)$ of a K-vector space V is an $\mathbb{N}$-graded associative algebra generated over $K = \bigwedge^0(V)$ by $V = \bigwedge^1(V)$ with multiplication denoted $\wedge$ satisfying precisely those relations implied by the condition that $v \wedge v = 0$ for every element $v \in V$. Then

$$0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w = v \wedge w + w \wedge v,$$

so that

$$v \wedge w = -w \wedge v$$

for all $v, w \in V$. This implies that if $\{v_j\}_{j \in \mathcal{J}}$ is an ordered basis for V then the elements $v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_i}$ such that $v_{j_1} > v_{j_2} > \cdots > v_{j_i}$ form a basis for $\bigwedge^i(V)$. In particular, it follows that if $\dim_K(V) = k$, then

$$\dim_K\big(\textstyle\bigwedge^i(V)\big) = \binom{k}{i}, \ 0 \le i \le k,$$

while $\bigwedge^i(V) = 0$ for $i > \dim(V)$.

Moreover, for any elements $v_1, \ldots, v_k \in V$, we have that for every permutation π of $\{1, \ldots, k\}$,

$$v_{\pi(1)} \wedge \cdots \wedge v_{\pi(k)} = \mathrm{sgn}(\pi)(v_1 \wedge \cdots \wedge v_k),$$

where $\mathrm{sgn}(\pi)$ is the sign of the permutation π. We also have that $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1, \ldots, v_k$ are linearly dependent over K. We know that if $v_1, \ldots, v_k$ is a basis for V, then the single element $v_1 \wedge \cdots \wedge v_k$ is a basis for $\bigwedge^k(V)$, which is a one-dimensional space. If we consider k linear combinations of $v_1, \ldots, v_k$, say

$$w_i = c_{i1}v_1 + \cdots + c_{ik}v_k$$

for $1 \le i \le k$, with the elements $c_{ij} \in K$, then

$$w_1 \wedge \cdots \wedge w_k = \det(c_{ij})\, v_1 \wedge \cdots \wedge v_k,$$

which will be another generator of $\bigwedge^k(V)$ precisely when $w_1, \ldots, w_k$ is a basis for V.

Note that forms of even degree in $\bigwedge^\bullet(V)$ are in the center, while if w, w' are forms of odd degree, $w \wedge w' = -w' \wedge w$. Notice also that, by definition, $\bigwedge^0(V) = K$. An $\mathbb{N}$-graded associative algebra such that for any two nonzero forms w, w' of degrees d, d' respectively, $ww' = (-1)^{dd'} w'w$ is called a *skew-commutative graded algebra*. (Some call such graded algebras *commutative*, but we shall not do this.)

If $T : V \to W$ is a K-linear map, it extends uniquely to a degree preserving K-homomorphism of $\mathbb{N}$-graded associative algebras $\bigwedge^\bullet(T) : \bigwedge^\bullet(V) \to \bigwedge^\bullet(W)$. This makes $\bigwedge^\bullet(\_)$ into a covariant functor from K-vector spaces and K-linear maps to skew-commutative graded K-algebras and degree-preserving K-algebra homomorphisms. In particular, we have functorial maps $\bigwedge^i(T) : \bigwedge^i(V) \to \bigwedge^i(W)$ for every $i \in \mathbb{N}$. Observe also that

$$T(v_1 \wedge \cdots \wedge v_k) = T(v_1) \wedge \cdots \wedge T(v_k).$$

If $W \subseteq V$ is a k-dimensional vector space, then $\bigwedge^k(W) \subseteq \bigwedge^k(V)$ is a one-dimensional subspace of $\bigwedge^k(V)$. This one dimensional subspace uniquely determines W, since if $\bigwedge^k(W) = Kw$ then $W = \{v \in V : w \wedge v = 0\}$.

Given an ordered basis for V, we introduce an order on the basis for $\bigwedge^k V$ mentioned above. A typical element of the basis for $\bigwedge^k V$ has the form $v_1 \wedge \cdots \wedge v_k$ where $v_1, \ldots, v_k$ are in the given ordered basis for V and are such that $v_1 > \cdots > v_k$. The ordering is given by the following rule: if $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are in this basis with $v_1 > \cdots > v_k$ and $w_1 > \cdots w_k$, we define $v_1 \wedge \cdots \wedge v_k > w_1 \wedge \cdots \wedge w_k$ to mean that there exists i, $1 \le i \le k$, such that $v_j = w_j$ for $j < i$ and $v_i > w_i$. This ordering resembles lexicographic ordering of monomials.

Remark. Suppose that $v_1, \ldots, v_k$ are distinct elements of the ordered basis, not necessarily in decreasing order, and that $w_1, \ldots, w_k$ are distinct elements of the ordered basis, also not necessarily in decreasing order. Suppose that for every i, $(*) \quad v_i \ge w_i$. Then this condition also holds when both sequences are arranged in decreasing order. The reason is simply this: let $v'_1, \ldots, v'_k$ and $w'_1, \ldots, w'_k$ denote the sequences arranged in decreasing order. For every i, each of the elements $w'_1, \ldots, w'_i$ is less than some element of $v_1, \ldots, v_k$ coming from the inequalities $(*)$, where these i elements are mutually distinct. Then w_i is less than or equal to each of these i distinct elements of the $v_1, \ldots, v_k$. The smallest of these i elements is evidently at most v'_i. $\square$

We then have:

Proposition. *Let V be a vector space with ordered basis, and let W be a subspace of dimension k. Then a reduced Gröbner basis for W is a basis for W, and given such a basis $w_1, \ldots, w_k$, we have that* $\mathrm{in}_{\mathrm{vec}}(w_1) \wedge \cdots \wedge \mathrm{in}_{\mathrm{vec}}(w_k)$ *is the initial term of a generator for* $\mathrm{in}_{\mathrm{vec}}\big(\bigwedge^k(W)\big)$.

Proof. Fix sufficiently many elements $v_1 > \cdots > v_s$ of the ordered basis for V so that W is contained in their span. We have already noted in the Lecture Notes of January 17 that the condition for $w_1, \dots, w_k$ to be a reduced Gröbner basis is that when each w_i is written in terms of $v_1, \dots, v_s$, the coefficients used, formed into the rows of a $k \times s$ matrix, produce a reduced row echelon matrix without any rows that are 0. (If any v_i are not actually used, they contribute columns that are entirely zero, and do not affect whether the matrix one obtains is in reduced row echelon form.) The leading entries of the rows correspond to the intial terms of the w_i. It is now clear from the Remark above that when we form $\omega_1 \wedge \cdots \wedge w_k$, the initial term is obtained by forming the product, under $\wedge$, of the initial terms. $\square$

We next observe that we can define a generic vector space of initial forms, $\mathrm{Gin}_{\mathrm{vec}}(W)$ when W is a k-dimensional subspace of V.

Theorem. *Fix a monomial order on $R = K[x_1, \dots, x_n]$: this yields an ordered basis for $\bigwedge^k(R)$ for all $k \in \mathbb{N}$. Let $W \subseteq R = K[x_1, \dots, x_n]$ be a given subspace of finite dimension. There is a Zariski open dense subset U of $\mathrm{GL}(n, K)$ such that $\mathrm{in}_{\mathrm{vec}}(AW)$ is the same for all $A \in U$. U may chosen so that $\mathcal{B}_n^{\mathrm{L}}U = U$. If g generates $\bigwedge^k(W)$ as a K-vector space, then $\mathrm{Gin}_{\mathrm{vec}}(g)$ is the greatest term occurring in Ag for any $A \in \mathrm{GL}(n, K)$.*

Proof. Let $Z = \bigl(z_{ij}\bigr)$ be a matrix of new indeterminates. Exactly as in our earlier proof of the existence of generic initial modules, we may consider ZW over $K(Z)$ and construct the reduced Gröbner basis, in the $K(Z)$-vector space sense, there, keeping track of finitely many polynomials in the z_{ij} that are used in denominators and also finitely many polynomials in the z_{ij} that occur as numerators of coefficients of initial terms. We may form the product P of these polynomials in the z_{ij}, and then we may take the set where P does not vanish as a choice of U. Applying a matrix in $\mathcal{B}^{\mathrm{L}}$ does not change the initial term of an element, and hence $\mathcal{B}_n^{\mathrm{L}}U$ is a larger dense open set for which every matrix yields the same initial vector space. Finally, note that a term that occurs in some Ag will occur in Zg. Since the leading term of Zg gives $\mathrm{Gin}_{\mathrm{vec}}(g)$, and is greater than any other term in Zg, the final statement follows. $\square$

We next want to characterize when a K-vector subspace of R is stable under the action of the diagonal matrices $\mathcal{D}_n$, and when such a subspace is stable under the action of the $\mathcal{B}_n^{\mathrm{U}}$. We first note that over an infinite field K, we have a graded vector space analogue of the Proposition on p. 4 of the Lecture Notes of February 9.

First note that if V is an $\mathbb{N}$ or $\mathbb{Z}$-graded vector space over an infinite field K, then we may define an automorphism η_u of V for each nonzero $u \in K$ such that each element $v \in [V]_d$ maps to $u^d v$.

Proposition. *Let V be an $\mathbb{N}$- or $\mathbb{Z}$-graded vector space over an infinite field K. Then every subspace W of V stable under all the η_u for $u \in K - \{0\}$ is graded.*

Proof. The proof is identical with the proof given for the earlier Proposition: the van Der Monde matrix Q now has entries in K, and W replaces the ideal I. $\square$

Theorem. *Let R be a polynomial ring $K[x_1, \dots, x_n]$, where K is a field. Let W be a K-vector subspace of R. Then W is stable under the action of $\mathcal{D}_n$ if and only if W is spanned by monomials.*

Suppose that K has characteristic p (which may be 0). Then W is stable under the action of $\mathcal{B}_n^{\mathrm{U}}$ if and only if it is spanned by monomials and for every monomial $\mu \in W$, if $\mu = x_j^k \nu$, where x_j does not divide μ, $h \leq_p k$, and $i \leq j$ then $x_i^h x_j^{k-h} \nu \in W$.

Proof. The proof of the first statement is identical with the proof of the Corollary on p. 5 of the Lecture Notes of February 9, using the Proposition above, and the proof of the second statement is identical with the proof of the Proposition on p. 7 of the Lecture Notes of February 9. $\square$

We refer to the subspaces of R stable under $\mathcal{B}_n^{\mathrm{U}}$ as *Borel-fixed*.

We next note that there is a monomial grading of $\bigwedge^\bullet(R)$. If one has terms $v_1, \dots, v_k$ involving mutually distinct monomials $\mu_1, \dots, \mu_k$, we define the *monomial degree* of the element $v_1 \wedge \cdots \wedge v_k$ to be the product $\mu_1 \cdots \mu_k$. If the μ_i are not mutually distinct, then $v_1 \wedge \cdots \wedge v_k = 0$. Let$[\bigwedge^\bullet(R)]_\mu$ denote the K-span of all basis elements whose monomial degree is μ. Then we have a direct sum decomposition

$$\bigwedge^\bullet(R) = \bigoplus_{\mu \in \mathcal{M}} [\bigwedge^\bullet(R)]_\mu,$$

where $\mathcal{M}$ is the set of monomials in R. Note that if $R = K[x_1, x_2]$ and $\mu = x_1^3 x_2^3$, then $[\bigwedge^\bullet(R)]_\mu$ contains $x_1^3 x_2^3$, $x_1^2 \wedge x_1 x_2^3$, and $x_1^2 \wedge x_1 x_2 \wedge x_2^2$, as well as many other elements.

Remark. A critical observation is the following: if W is a k-dimensional K-vector subspace of R and w generates $\bigwedge^k(W)$, then the monomial degree of $\mathrm{in}(w)$ is strictly larger than the monomial degree of any other term of w. Consider a Gröbner basis for W as a vector space: in each element, the initial monomial is strictly larger than any other monomial occurring. The product of the initial monomials is therefore strictly larger than the product of any other choice of monomials, one from each factor, from which the assertion follows.

The action of $\mathcal{D}_n$ on R induces an action on $\bigwedge^\bullet(R)$. Note that if $\beta = (b_1, \dots, b_n) \in (K - \{0\})^n$, and $\mathrm{diag}(b_1, \dots, b_n)$ is the diagonal matrix with b_i in the i, i position on the main diagonal for $1 \leq i \leq n$, then for any element $v \in [\bigwedge^\bullet(R)]_\mu$, we have that $Bv = \mu(\beta)v$, where $\mu(\beta)$ denotes the result of substituting $x_1 = b_1, \dots, x_n = b_n$ in μ.

Theorem. *Let $W \subseteq R$ be any finite dimensional vector space. Let Then* $\mathrm{Gin}_{\mathrm{vec}}(W)$ *is Borel-fixed.*

Proof. First replace W by AW such that $\mathrm{in}_{\mathrm{vec}}(AW) = \mathrm{Gin}_{\mathrm{vec}}(W)$. Let w generate the one-dimensional vector space $\bigwedge^k(W)$. It suffices to show that for every upper triangular elementary matrix $E = E_{ij}(c)$, $E\big(\mathrm{in}_{\mathrm{vec}}(w)\big) = \mathrm{in}_{\mathrm{vec}}(w)$. The action of E on a monomial term produces a linear combination of monomial terms one of which is the original term, while the others are strictly larger — if the term has the form $x_j^k \nu$ where ν is a term not divisible by x_j, this this follows from the expansion

$$(cx_i + x_j)^k \nu = \sum_{0 \le h \le_p k} \binom{k}{h} c^k x_i^h x_j^{k-h} \nu,$$

which was used in the proof of the Proposition on p. 7 of the Lecture Notes of February 9. It follows from the Remark on p. 3 that if $E\big(\mathrm{in}_{\mathrm{vec}}(w)\big) \neq \mathrm{in}_{\mathrm{vec}}(w)$, all of its nonzero terms other than $\mathrm{in}_{\mathrm{vec}}(w)$ are larger than $\mathrm{in}_{\mathrm{vec}}(w)$. Pick one such term τ. It suffices to show that τ survives in $EB(w)$ for some upper triangular matrix $B = \mathrm{diag}(b_1, \ldots, b_n)$ where $\beta = (b_1, \ldots, b_n) \in (K - \{0\})^n$. Let μ be the monomial degree of $\mathrm{in}(w)$. By the Remark just above, the monomial degree of every other term in w is strictly smaller than μ, so that

$$w = \mathrm{in}(w) + \sum_{\nu < \mu} w_\nu.$$

Then

$$EB(w) = E(B\mathrm{in}(w)) + \sum_{\nu < \mu} E(B(w_{nu}) = \mu(\beta) E\big(\mathrm{in}(w)\big) + \sum_{\nu < \mu} \nu(\beta) E(w_{nu}).$$

Consider the coefficient of τ in the final expression on the right as a function of β. The first sumand makes a contribution $\mu(\beta)$ to this coefficient. The other contributions to the sum have the form $c_\nu \nu(\beta)$ for $\nu < \mu$. It follows that the coefficient of τ is a nonzero polynomial in β, since the $\mu(\beta)$ term cannot be canceled. Hence τ occurs in $EB(w)$ for some choice of B, which contradicts the last statement in the Theorem stated on the bottom of p. 3 and the top of p. 4. $\square$