Math 615: Lecture of February 14, 2007

We postpone further consideration of Gröbner bases to study some results in invariant theory.

To keep prerequisites from algebraic geometry to a minimum, in our study we will take the ground field K to be an algebraically closed field. For the kinds of results that we will be considering, this is no disadvantage: typically, one can deduce results over any infinite field by passing to the algebraic closure.

Linear algebraic groups and their modules

We have seen that $\mathrm{GL}(n, K)$ has the structure of a closed algebraic set, and that the same is true for the $\mathrm{GL}_n(V)$, the group of K-automorphisms of a finite-dimensional vector space V. See pages 1. and 2. of the Lecture of January 31. One gives $\mathrm{GL}_n(V)$ the structure of a closed algebraic set by choosing a basis for V. If $\dim(V) = n$, this gives an identification of V with $\mathrm{GL}(n, K)$. However, the structure of V as an algebraic set is independent of the choice of basis: if one takes a different basis, the identification of $GL(n, K)$ with V changes, but this is via an automorphism of $\mathrm{GL}(n, K)$ given by conjugating by the change of basis matrix. This map is not only a group automorphism: it is also an automorphism in the category of closed algebraic sets.

A *linear algebraic group* G is a Zariski closed subgroup of some $\mathrm{GL}(n, K)$. Thus, G has the structure of closed algebraic set.

The product of two closed algebraic sets has the structure of a closed algebraic set. If $X = V(I)$ where $I \subseteq K[x_1, \ldots, x_m]$, so that $X \subseteq \mathbb{A}^m_K$, and $Y = V(J)$ where $J = K[y_1, \ldots, y_n]$, so that $Y \subseteq \mathbb{A}^n_K$ (the variables are taken to be $m + n$ algebraically independent elements) then $X \times Y$ may be identified with $V(IT + JT) \subseteq \mathbb{A}^{m+n}_K$, where $T = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

It is easy to show that if G is a linear algebraic group, then the map $G \times G \to G$ that corresponds to the group multiplication is regular, as well as the inverse map $G \to G$: this follows from the fact that this is true when $G = \mathrm{GL}(n, K)$.

An *action* of a linear algebraic group G on a finite-dimensional vector space V is then a group action $G \times V \to V$ such that the defining map is a morphism of closed algebraic sets, i.e., a regular map over K. The image of (γ, v) is denoted $\gamma(v)$. Alternatively, it is given by a homomorphism $h : G \to GL_K(V)$: the action is recovered by the rule $\gamma(v) = h(\gamma)(v)$. We then say that V is G-module (over K, but usually we do not mention the field K).

If $W \subseteq V$ is a K-vector subspace such that W is stable under the action of G, the restriction of the map $G \times V \to V$ gives W the structure of a G-module, and we shall say that W is a G-*submodule* of V.

We extend the notion of G-module to infinite-dimensional K-vector spaces as follows: an action of G on an infinite-dimensional vector space V is allowed if V is a directed union of finite-dimensional spaces W such that the restricted action makes W into a G-module.

The direct sum of G-modules becomes a G-module in an obvious way. A G-stable subspace of an infinite-dimensional G-module is again a G-module. If V is a G-module and $W \subseteq V$, then V/W has the structure of G-module such that for all $\gamma \in G$ and $v \in V$, $\gamma(v + W) = \gamma(v) + W$.

A G-module map $f : V \to W$ is a K-linear map such that for all $\gamma \in G$ and $v \in V$, $f\big(\gamma(v)\big) = \gamma\big(f(v)\big)$. The inclusion of a G-submodule $W \subseteq V$ is a G-module map, as is the quotient map $V \twoheadrightarrow V/W$.

A nonzero G-module M is called *irreducible* or *simple* if it has no nonzero proper submodule. If M is irreducible it is necessarily finite-dimensional, as it is a directed union of finite-dimensional G-submodules.

A linear algebraic group is called *linearly reductive* if every finite-dimensional G-module is a direct sum of irreducible G-modules. Over an field, the finite groups G such that the order of G is invertible in the field are linearly reductive, and so is an algebraic torus, i.e., a finite product of copies of $GL(1,\, K)$. In characteristic $p > 0$, these are the main examples. But over $\mathbb{C}$ the semisimple groups are linearly reductive as well. We shall comment further about this later.

Linearly reductive linear algebraic groups

Theorem. *Let G be a linearly reductive linear algebraic group and let $W \subseteq V$ be G-modules. Then there is a family of irreducible submodules $\{M_\lambda\}_{\lambda\in\Lambda}$ in V such that*

$$V = W + \sum_{\lambda\in\Lambda} M_\lambda$$

and the sum is direct. Hence, if

$$W' = \sum_{\lambda\in\Lambda} M_\lambda,$$

then $V = W \oplus W'$, so that W' is a G-module complement for W in V.

In particular, we may take $W = 0$, and so V itself is a direct sum of irreducible submodules, even if it is infinite-dimensional.

Proof. Consider the set of families of irreducible submodues

$$\{M_\lambda\}_{\lambda\in\Lambda}$$

of V such that the sum

$$W + \sum_{\lambda\in\Lambda} M_\lambda$$

is direct, i.e., such that every module occurring has intersection 0 with the sum of the other modules occurring. The empty set is such a family, and the union of chain of such families is such a family. Hence, there is a maximal such family, which we denote $\{M_\lambda\}_{\lambda\in\Lambda}$. We claim that $V = V'$, where

$$V' = W + \sum_{\lambda\in\Lambda} M_\lambda.$$

If not, there is a finite-dimensional submodule V_0 of V that is not contained in V'. V_0 is a direct sum of irreducibles: one of these, call it M_0, must also fail to be contained in V'. Then $M_0 \cap V'$ is a proper G-submodule of M_0, and so it is 0. But then the family can be enlarged by including M_0 as a new member, a contradiction. □

If V is G-module, let V^G be the *subspace of invariants*, i.e.,

$$V^G = \{v \in V : \text{for all } \gamma \in G,\ \gamma(v) = v\}.$$

Then V^G is the largest G-submodule of V on which G acts trivially, and it is a direct sum (although not in a unique way) of one-dimensional G-modules on which G acts trivially. Note that if M is an irreducible G-module on which G acts on non-trivially, then $M^G = 0$, for otherwise M^G is a proper nonzero G-submodule of M.

Theorem. *Let V be a G-module, where G is linearly reductive. Then V^G has a* unique *G-module complement V_G, which may also be characterized as the sum of all irreducible submodules M of V on which G acts non-trivially.*

Proof. Let W be any G-module complement for V^G. Let M be any irreducible in G on which G acts non-trivially. If $M \cap W \neq 0$, the $M \cap W = M$, and so $M \subseteq W$ as required. Otherwise M injects into $V/W \cong V^G$, which implies that G acts trivially on M, a contradiction. Thus, every irreducible on which G acts nontrivially is contained in W. But W is a direct sum of irreducibles, and G must act non-trivially on each of these, since there are no invariants in W. Therefore, W is the sum of all irreducible submodules of G on which G acts non-trivially, which proves that W is unique. □

We also have:

Proposition. *If $f : V \to W$ is a map of G-modules, then $f : V^G \to W^G$, i.e., f induces a map of the respective G-invariant subspaces of V and W by restriction. Moreover, $f : V_G \to W_G$. Thus, f preserves the direct sum decompositions $V = V^G \oplus V_G$ and $W = W^G \oplus W_G$.*

Proof. If v is invariant so that $\gamma(v) = v$ for all $\gamma \in G$, then $\gamma\big(f(v)\big) = f\big(\gamma(v)\big) = f(v)$ for all $\gamma \in G$. Thus, $F(V^G) \subseteq W^G$.

Now consider any irreducible M on which G acts non-trivially. The kernel of f intersected with M is a G-submodule of M, and, hence, is 0 or M. If it is 0, then M injects into W, and the image is an isomorphic copy of M, which means that $f(M)$ is an irreducible

G-submodule of W on which G acts non-trivially. Hence, $f(M) \subseteq W_G$. On the other hand, if the kernel contains all of M, the image is $0 \subseteq W_G$. $\square$

Dicussion. Let G be a linear algebraic group that is not necessarily lineaarly reductive. Consider a short exact sequence of G-modules

$$0 \to W \to V \to Y \to 0.$$

Clearly, $W^G \subseteq Y^G$, and the kernel of the map $V^G \to Y^G$ is, evidently, $V^G \cap W$, which is obviously W^G. Hence, for any linear algebraic group, we always have that

$$0 \to W^G \to Y^G \to V^G$$

is exact. In general, however, the map $Y^G \to V^G$ need not be onto. However:

Corollary. *If G is linearly reductive and $0 \to W \to V \to Y \to 0$ is an exact sequence of G-modules, then $0 \to W^G \to V^G \to Y^G \to 0$ is exact.*

Proof. The map $V \to Y$ is the direct sum of the maps $V^G \to Y^G$ and $V_G \to Y_G$. Hence, it is surjective if and only if both $V^G \to Y^G$ and $V_G \to Y_G$ are surjective, which, in particular, shows that $V^G \to Y^G$ is surjective. $\square$

When G is linearly reductive, we have a canonical G-module retraction $\rho_V : V \twoheadrightarrow V^G$ that is obtained by killing V_G. This map is called the *Reynolds operator.* Note that if we are given a short exact sequence of G-modules $0 \to W \to Y \to V \to 0$, then we have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_G & \longrightarrow & V_G & \longrightarrow & Y_G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & V & \longrightarrow & Y & \longrightarrow & 0 \\
 & & {\scriptstyle \rho_W}\downarrow & & {\scriptstyle \rho_V}\downarrow & & {\scriptstyle \rho_Y}\downarrow & & \\
0 & \longrightarrow & W^G & \longrightarrow & V^G & \longrightarrow & Y^G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The columns are split exact, and the rows are exact: the middle row is the direct sum of the rows above and below it.

The property that when $V \to W$ is a surjection of finite-dimensional G-modules then $V^G \to W^G$ is surjective actually characterizes linearly reductive groups. To see this, first note that if V and W are finite-dimensional G-modules, we can put a G-module structure on $\mathrm{Hom}_K(V, W)$ (this is simply the vector space of all K-linear maps) as follows: for all $\gamma \in G$ and all $f : V \to W$, $\gamma(f)(v) = \gamma(f(\gamma^{-1}v)$. This is easily verified to give $\mathrm{Hom}_K(V, W)$ the structure of a G-module. Moreover:

Lemma. *Let V, W be finite-dimensional G-modules. Then $\mathrm{Hom}_K(V, W)^G$ is the K-vector space of G-module maps from V to W.*

Proof. Suppose that $f : V \to W$. Then f is fixed by G if and only if for all $\gamma \in G$ and for all $v \in V$, $\gamma\big(f(\gamma^{-1}v)\big) = f(v)$, i.e., $f(\gamma^{-1}v) = \gamma^{-1}f(v)$. Since γ^{-1} takes on every value in G as γ varies, we have that f is fixed by G iff f is a G-module homomorphism. □

Theorem. *Let G be a linear algebraic group. G is linearly reductive if and only if for every surjective G-module map of finite-dimensional G-modules $V \twoheadrightarrow W$, the map $V^G \to W^G$ is also surjective.*

Proof. It suffices to show that every finite-dimensional G-module V is a direct sum of irreducible G-modules: if not, let V be a counter-example of smallest possible vector space dimension. Then V is not irreducible, and we may choose a maximal proper G-submodule $M \neq 0$, so that $W = V/M$ is irreducible. It suffices to show that the exact sequence

$$(*) \quad 0 \to M \to V \xrightarrow{f} W \to 0$$

splits as a sequence of G-modules, since in that case we have that $V \cong M \oplus W$ and $\dim_K(M) < \dim_K(V)$. It is, of course, split as a sequence of K-vector spaces. Apply $\mathrm{Hom}_K(W, \_)$, where this is simply Hom as K-vector spaces. Then

$$0 \longrightarrow \mathrm{Hom}_K(W, M) \longrightarrow \mathrm{Hom}_K(W, V) \xrightarrow{f_*} \mathrm{Hom}_K(W, W) \longrightarrow 0$$

is exact (since the sequence $(*)$ is split as a sequence of K-vector spaces), and the map f_*, which sends $g : W \to V$ to $f \circ g$, is therefore surjective. This is a sequence of G-modules, and so the map

$$\mathrm{Hom}_K(W, V)^G \to \mathrm{Hom}_K(W, W)^G$$

is surjective. That is, the set of G-module maps from $W \to V$ maps onto the set of G-module maps from $W \to W$. Hence, there is a G-module map $g : W \to V$ such that $f_*(g) = f \circ g$ is the identity map on W, and so $(*)$ is split as a sequence of G-modules. □

Remark. The existence of a functorial Reynolds operator that retracts every finite-dimensional G-module onto its invariant submodule and so, for every G-module map $V \to W$, provides a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & > W \\ {\scriptstyle \rho_V}\big\downarrow & & \big\downarrow{\scriptstyle \rho_W} \\ V^G & \xrightarrow[\quad f \quad]{} & W^G \end{array}$$

already implies that when the top arrow is surjective, so is the bottom arrow. For if $w \in W^G$ we may choose an arbitrary element $v \in V$ such that $f(v) = w$, and then

$$f\big(\rho_V(v)\big) = \rho_W\big(f(v)\big) = \rho_G(w) = w,$$

as required. Thus, the existence of a functorial retraction onto the modules of invariants is also equivalent to the condition that G be linearly reductive.

Remark. If G is a finite group such that the order $|G|$ of G is invertible in K, the Reynolds operator is given by:

$$\rho(v) = \frac{1}{|G|} \sum_{g \in G} g(v),$$

i.e., averaging over the group G.

It turns out that linear reductive linear algebraic groups over the complex numbers $\mathbb{C}$ are precisely those that have a Zariski dense compact real Lie subgroup H. Then H has Haar measure, a translation-invariant measure μ such that $\mu(H) = 1$, and the Reynolds operator can be obtained by averaging over the group:

$$\rho(v) = \int_{\gamma \in H} \gamma(v)\, d\mu.$$

Early proofs of finite generation for rings of invariants of semisimple groups over $\mathbb{C}$ made use of this idea. Purely algebraic proofs have been available for a long time: these involve the study of modules over the Lie algebra. See, for example, [A. Borel, *Linear Algebraic Groups*, Benjamin, New York, 1969].