Math 615: Lecture of March 12, 2007

Tight closure for modules

We want to extend tight closure theory to modules. Suppose we are given $N \subseteq M$, finitely generated modules over a Noetherian ring R of prime characteristic $p > 0$. We can define v^{p^e} for $v \in R^h$ as follows: if $v = (f_1, \ldots, f_h)$, then $v^{p^e} = (f_1^{p^e}, \ldots, f_h^{p^e})$. If $G \subseteq R^h$ we define G^{p^e} as the R-span of all the elements $\{v^{p^e} : v \in G\}$. One gets the same module if one takes only the R-span of the p^e th powers of generators of G. This agrees with our definition of $I^{[p^e]}$ when $I \subseteq R$ is an ideal. If $G \subseteq R^h$, we define $G^*_{R^h}$, the *tight closure* of G in R^h as the set of elements $v \in R^h$ such that for some $c \in R^\circ$, $cv^q \in G^{[q]}$ for all $q \gg 0$, where q is p^e.

Given $N \subseteq M$ where M is finitely generated over R, we define the *tight closure* N^*_M of N in M as follows. Map a free module $R^h \twoheadrightarrow M$, and let G be the inverse image of N in R^h, so that we also have a surjection $G \twoheadrightarrow N$. Let v be any element of R^h that maps to u. Then $u \in N^*$ precisely if $v \in G^*_{R^h}$ as defined above. This is independent of the choice of v mapping to u. It is also independent of the choice of surjection $R^h \twoheadrightarrow M$.

It is understood that the tight closure of an ideal is taken in R unless otherwise specified.

Note that:

(0) $u \in N^*_M$ *if and only if the image* $\overline{u}$ *of* u *in* M/N *is in* $0^*_{M/N}$.

As in the ideal case:

(1) N^*_M *is a submodule of* M *and* $N \subseteq N^*_M$. *If* $N \subseteq Q \subseteq M$ *then* $N^*_M \subseteq Q^*_M$.

(2) If $N \subseteq M$, then $(N^*_M)^*_M = N^*_M$.

An example of tight closure

Let K be any field of characteristic $p > 0$ with $p \neq 3$. Let

$$R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z].$$

This is a normal ring with an isolated singularity. It is Cohen-Macaulay. It is also a standard graded K-algebra. (This ring is sometimes called a *cubical cone*. It is also the homogeneous coordinate ring of an elliptic curve.)

We claim that $z^2 \in (x, y)^* - (x, y)$ in R. In fact, if we kill $I = (x, y)R$, we have $R/I = K[Z]/(Z^3)$, and the image of Z^2 is not 0. Take $c = z$ (the choices $c = x$ and $c = y$ also work). We need to check that

$$z(z^{2q}) \in (x^q, y^q)$$

for all $q \gg 0$. Let ρ be the remainder when $2q+1$ is divided by 3, so that $\rho = 0$ or $\rho = 2$. We can write $2q+1 = 3k+\rho$. Then

$$c(z^2)^q = z^{2q+1} = z^{3k+\rho} = (z^3)^k z^\rho = (-1)^k (x^3+y^3)^k z^\rho.$$

To conclude the proof that $z^2 \in (x,\, y)^*$, it suffices to show that $(x^3+y^3)^k \in (x^q,\, y^q)$. But otherwise we have $i+j=k$ with $i \geq 0$ and $j \geq 0$, and this implies that $3i \leq q-1$ and that $3j \leq q-1$. Adding these inequalities gives $3k = 3i+3j \leq (q-1)+(q-1) = 2q-2$, so that $2q+1-\rho \leq 2q-2$ which implies that $\rho \geq 3$, a contradiction. $\square$

This gives a non-trivial example where the tight closure of an ideal is larger than the ideal.

Defining tight closure for Noetherian rings containing the rational numbers

We want to discuss very briefly how one extends the theory to all Noetherian rings containing $\mathbb{Q}$. For a detailed account see, [M. Hochster and C. Huneke, *Tight closure in equal characteristic zero*, preprint] available at

`http://www.math.lsa.umich.edu/~hochster/msr.html`

— the notion discussed here corresponds to ${}^{*\mathrm{eq}}$. There is also an exposition in [M. Hochster, *Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure*, in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra*, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 1994, 173–196].

We first define a notion of tight closure in finitely generated $\mathbb{Q}$-algebras. In fact, any finitely generated $\mathbb{Q}$-algebra can be obtained as the tensor product over $\mathbb{Z}$ of $\mathbb{Q}$ with a finitely generated $\mathbb{Z}$-algebra. If our original $\mathbb{Q}$ algebra is $R = \mathbb{Q}[X_1,\, \ldots\,, X_n]/(F_1,\, \ldots\,, F_m)$, note that one can choose a single integer d divisible by all denominators in the polynomials $F_1,\, \ldots\,, F_m$, and then

$$R = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[1/d][X_1,\, \ldots\,, X_n]/(F_1,\, \ldots\,, F_m).$$

We want to keep track of the behavior of this finitely generated $\mathbb{Z}$-algebra as we localize at finitely many nonzero integers: of course, this has the same effect as localizing $\mathbb{Z}$ at a single nonzero integer. Therefore we shall think of our finitely generated $\mathbb{Q}$-algebra R as $\mathbb{Q} \otimes_D R_D$, where $D = \mathbb{Z}[1/d]$ is the localization of $\mathbb{Z}$ at a single nonzero integer. But we shall allow that integer d to change so that it has more factors: in effect, as we localize further, we exclude finitely many more prime integers from consideration. By localizing at one element of $\mathbb{Z} - \{0\} \in D$ we may assume that R_D is D-free, by the Theorem on generic freeness. If B is D-algebra, which typically will be either $\mathbb{Q}$ or $\kappa = D/pD$ for some prime integer $p > 0$ not invertible in D, we write R_B for $B \otimes_D R_B$. Thus, $R = R_{\mathbb{Q}}$. Moreover, if M_D is an R_D-module, we write M_B for $B \otimes_D M_D$.

Given a finitely generated R-module M, we may think of it as the cokernel of a finite matrix with entries in D. This matrix will have entries in R_D if we localize D sufficiently, so that we have an R_D-module M_D such that $\mathbb{Q} \otimes_D M_D \cong M$. If D is large enough, we can assume that a given element of M is in D. If N is a finitely generated submodule of M, we may assume that D is large enough to contain a given finite set of generators of N over R, and we consider the R_D-submodule N_D of M_D generated by these elements. By localizing D at one more nonzero integer, we may assume that all of the terms of

$$0 \to N_D \to M_D \to M_D/N_D \to 0$$

are D-free. It follows that

$$0 \to N_B \to M_B \to M_B/N_B \to 0$$

is exact for every D-algebra B. We then have that $N \subseteq M$ arises from the inclusion $N_D \subseteq M_D$ by applying $\mathbb{Q} \otimes_D \_$. Note that when $M = R$ and $N = I$ is an ideal of R, we localize so that R_D/I_D is D-free.

Now suppose whether we want to test whether $u \in M$ is in the tight closure of N in M in *the affine* $\mathbb{Q}$*-algebra* sense. We choose R_D and $N_D \subseteq M_D$ as above, and take D sufficiently large that $u \in M_D$. We then define $u \in N_M^*$ if the image of $1 \otimes u$ of u is in $N_\kappa^* \subseteq M_\kappa$, where $\kappa = D/pD = \mathbb{Z}/p\mathbb{Z}$, for all but finitely many prime integers $p > 0$ that are prime in D. This condition can be shown to be independent of the choice of D, R_D, and $N_D \subseteq M_D$. This turns out to give a very good notion of tight closure when the base ring is a finitely generated $\mathbb{Q}$-algebra.

Example. Consider $R = \mathbb{Q}[X, Y, Z]/(X^3 + Y^3 + Z^3) = \mathbb{Q}[x, y, z]$. Then in this ring we have $z^2 \in (x, y)^*$, just as we did in positive characterisitic $p \neq 3$. In fact, we can take $D = \mathbb{Z}$ and $R_D = \mathbb{Z}[X, Y, Z]/(X^3 + Y^3 + Z^3)$. We can let $I_D = (x, y)R_D$. For every $p \neq 3$, with $\kappa = \mathbb{Z}/p\mathbb{Z}$, the image of z^2 in $R_\kappa = \kappa[X, Y, Z]/(X^3 + Y^3 + Z^3)$ is in the tight closure, in the characteristic $p > 0$ sense, of $I_\kappa = (x, y)_\kappa$.

This notion can be extended to arbitrary Noetherian rings containing $\mathbb{Q}$ as follows. Let S be any such ring, let M be a finitely generated S-module and $N \subseteq M$ a submodule. Let $u \in M$. Then we define $u \in N_M^*$ if for every map $S \to C$, where C is a complete local domain, there exists and affine $\mathbb{Q}$-algebra R_0, a finitely generated R_0-module M_0, a submodule $N_0 \subseteq M_0$, an element $u_0 \in M_0$, and a map $R_0 \to C$ such that:

(1) $C \otimes_{R_0} M_0 \cong C \otimes_S M$.

(2) The image of $C \otimes_{R_0} N_0$ in $C \otimes_{R_0} M_0 \cong C \otimes_S M$ is the same as the image of $C \otimes_S N$ in $C \otimes_S M$.

(3) The image $1 \otimes u_0$ of u_0 in $C \otimes_{R_0} M_0 \cong C \otimes_S M$ is the same as the $1 \otimes u$ of u in $C \otimes M$.

(4) The element u_0 is in the tight closure of N_0 in M_0 in the affine $\mathbb{Q}$-algebra sense.

That, is roughly speaking, u is in the tight closure of $N \subseteq M$ if for every base change to a complete local domain, the new u, N, M also arise by base change from an instance of tight closure over an affine $\mathbb{Q}$-algebra.

This is a highly technical, convoluted definition, and working with it presents substantial technical difficulties. Nonetheless, with the help of some very deep results about the behavior of complete local rings, including a form of the Artin Approximation Theorem, one can show that this notion satisfies the conditions (1) — (5) discussed in the Lecture Notes for March 9 for a "good" tight closure theory. For the colon-capturing property (4) it suffices if the local ring is an *excellent* domain: we shall not define the property of being excellent here, but all rings that are localizations of finitely generated algebras over either a complete local ring (fields are included) or over $\mathbb{Z}$ are excellent.

We shall not pursue these ideas further in this course, but this should give the reader some feeling for how one extends the theory to all Noetherian rings containing $\mathbb{Q}$ in a manner that ultimately rests on reduction to characteristic $p > 0$.

Another use of tight closure: contracted expansions from module-finite extension rings

Let R be a domain. Suppose that $R \subseteq S$ is a module-finite extension. In general, $I \subseteq IS \subseteq R$, but $IS \cap R$ may be larger than I. The main case is where S is also a domain. For S has a minimal prime $\mathfrak{p}$ disjoint from the multiplicative system $R - \{0\}$, and R injects into $\overline{S} = S/\mathfrak{p}$, which is a domain module-finite over R. Moreover, if $r \in R$ is in IS, then the image of r in $S/\mathfrak{p}$ is in $I\overline{S}$.

Suppose that $f \in R$, $g \in R - \{0\}$, and f/g is integral over R but not in R, which means that $f \notin gR$. We may take $S = R[f/g]$. Then $f \in gS \cap R - gR$, so that when R is not normal even principal ideals fail to be contracted from module-finite extensions. But if R is normal and contains $\mathbb{Q}$, then every ideal is contracted from every module-finite extension S. To see this, first note that it suffices to consider the case where S is a domain, by the argument above. Let $\mathcal{K}$ and $\mathcal{L}$ be the respective fraction fields of R and S. Multiplication by an element of $\mathcal{L}$ gives a map $\mathcal{L} \to \mathcal{L}$ which is $\mathcal{K}$-linear. If we simply think of this map as an endomorphism of the finite-dimensional $\mathcal{K}$-vector space $\mathcal{L}$, we may take its trace: i.e., pick a basis for $\mathcal{L}$ over $\mathcal{K}$, and take the sum of the diagonal entries of the matrix of the multiplication map with respect to this basis. This is independent of the choice of basis.

This trace map $\mathrm{Tr}_{\mathcal{L}/\mathcal{K}} : \mathcal{L} \to \mathcal{K}$ is $\mathcal{K}$-linear (hence, R-linear) and has value h on 1, where $h = [\mathcal{L} : \mathcal{K}]$. When R is a normal Noetherian ring, it turns out that the values of this map on S are in R. (One can see this as follows. First, R is the intersection of its localizations R_P at height one primes P. For if $f,\, g \in R$, $g \neq 0$, and f/g is in the fraction field of R but not in R, then $f \notin gR$. The associated primes of gR have height one, because R is normal. Using the primary decomposition of gR, we see that $f \notin \mathfrak{A}$ for some ideal $\mathfrak{A}$ primary to an associated P of gR of height one, and since elements of $R - P$ are not zerodivisors on $\mathfrak{A}$, $f \notin \mathfrak{A}R_P$ and so $f \notin gR_P$, i.e., $f/g \notin R_P$. If $Tr_{\mathcal{L}/\mathcal{K}}$ has a value on S not in R, we may preserve this while localizing at a height one prime P of R. But then we may replace R, S by R_P, S_P and assume that $R = R_P$ is a Noetherian discrete valuation ring. Since S is a torsion-free module over R, it is free, and has a free basis over R, say $s_1, \ldots, s_j$, consisting of elements of S. This is also a basis for $\mathcal{L}$ over $\mathcal{K}$, and can be used to calculate the trace of s. But now the matrix for multiplication by s has entries in R: for every s_i we have

$$ss_i = \sum_{j=1}^{h} r_{ij}s_j$$

with the $r_{ij} \in R$. But then the trace is $\sum_{i=1}^{h} r_{ii}$ and is in R after all. The condition that R be Noetherian is not really needed: for example, in the general case, an integrally closed domain can be shown to be a directed union of Noetherian integrally closed domains, from which the general case can be deduced. There are several other lines of argument.)

Finally, $\dfrac{1}{h}\mathrm{Tr}_{\mathcal{L}/\mathcal{K}} : S \to R$ splits $R \hookrightarrow S$ as a map of R-modules: by R-linearity, the fact that 1 maps to itself implies that the same holds for every element of R. Since we have a splitting, it follows that every ideal of R is contracted from S.

Although ideals are contracted from module finite-extensions of normal Noetherian domains that contain $\mathbb{Q}$, this is false in positive characteristic p.

Example. Let $R = K[X, Y, Z]/(X^3+Y^3+Z^3)$ where K is a field of characteristic 2. Then $z^2 \notin (x, y)R$, as noted earlier. But if we make a module-finite domain extension S of R that contains $x^{1/2}$, $y^{1/2}$, and $z^{1/2}$, then since $z^3 = x^3 + y^3$ (we are in characteristic 2, so that minus signs are not needed) we have $z^{3/2} = x^{3/2} + y^{3/2}$ (since squaring commtutes with addition and elements have at most one square root in domains of characteristic 2, taking square roots also commutes with addition in domains of characteristic 2). But then

$$z^2 = z^{1/2}z^{3/2} = z^{1/2}(xx^{1/2} + yy^{1/2}) = x^{1/2}z^{1/2}x + y^{1/2}z^{1/2}y \in (x, y)S \cap R - R.$$

However, tight closure "captures" the contracted expansion to a module-finite extension, which gives another proof that $z^2 \in (x, y)^*$ in the Example just above.

Theorem. *Let R be a Noetherian domain, and let S be any integral extension of R. Then for every ideal I of R, $IS \cap R \subseteq I^*$.*

Proof. Suppose that $f \in R$ and

$$(*) \quad f = \sum_{i=1}^{h} f_j s_j$$

where the $f_j \in I$ and the $s_j \in S$. We may replace S by $R[s_1, \dots, s_h] \subseteq S$, and so assume that S is module-finite over R. Second, we may kill a minimal prime of S disjoint from $R-\{0\}$ and so assume that S is a module-finite domain extension of R. Choose a maximal set of R-linearly independent elements of S, say $u_1, \dots, u_k$, so that $Ru_1 + \cdots + Ru_k$ is R-torsion. It follows that some nonzero element $r \in R$, we have that

$$S \cong rS \subseteq Ru_1 + \cdots + Ru_k.$$

Thus, we have an embedding $S \hookrightarrow R^k$. Suppose that $1 \in S$ has as its image in R^k an element whose i th coordinate is nonzero, so that the composite map $S \hookrightarrow R^k \xrightarrow{\pi_i} R$ is nonzero on the element $1 \in S$, where π_i is the i th coordinate projection of $R^k \twoheadrightarrow R$. This gives an R-linear map $\theta : S \to R$ such that $\theta(1) = c \in R$ is nonzero. Now take q th powers of both sides of $(*)$, yielding

$$(**) \quad f^q \cdot 1 = \sum_{i=1}^{h} f_j^q s_j^q.$$

Since θ is R-linear and $f,\, f_1,\, \dots\, ,f_h \in R$, this yields

$$f^q \theta(1) = \sum_{i=1}^{h} f_j^q \theta(s_j^q),$$

and so $cf^q \in I^{[q]}$ for all q. This implies that $f \in I^*$. $\square$