Math 615: Lecture of March 14, 2007

Open questions: tight closure, plus closure, and localization

We want to consider some open questions in tight closure theory, and some related problems about when rings split from their module-finite extension algebras. After we do this, we shall prove some specific results in the characteristic p theory. It will turn out that to proceed further, we will need the structure theory of complete local rings, which we will develop next.

One of the longest standing and most important questions about tight closure is whether tight closure commutes with localization. E.g., if R is Noetherian of prime characteristic $p > 0$, I is an ideal of R, and W is a multiplicative system of R, is $W^{-1}(I^*_R)$ the same as $(W^{-1})^*_{W^{-1}R}$? It is easy to prove that $W^{-1}(I^*_R) \subseteq (W^{-1})^*_{W^{-1}R}$. This has been open question for more than twenty years. It is known to be true in many cases. See, for example, [I. Aberbach, M. Hochster, and C. Huneke, *Localization of tight closure and and modules of finite phantom projective dimension*, J. Reine Angew. Math. (Crelle's Journal) **434** (1993), 67–114], and [M. Hochster and C. Huneke, *Test exponents and localization of tight closure*, Michigan Math. J. **48** (2000), 305–329] for a discussion of the problem.

We saw in the last Theorem of the Lecture Notes of March 12 that tight closure "captures" contracted extension from module-finite and even integral extensions. We shall add this as (6) to our list of desirable properties for a tight closure theory, which becomes the following:

(0) $u \in N^*_M$ *if and only if the image* $\overline{u}$ *of* u *in* M/N *is in* $0^*_{M/N}$.

(1) N^*_M *is a submodule of* M *and* $N \subseteq N^*_M$. *If* $N \subseteq Q \subseteq M$ *then* $N^*_M \subseteq Q^*_M$.

(2) If $N \subseteq M$, then $(N^*_M)^*_M = N^*_M$.

(3) *If* $R \subseteq S$ *are domains, and* $I \subseteq R$ *is an ideal,* $I^* \subseteq (IS)^*$, *where* I^* *is taken in* R *and* $(IS)^*$ *in* S.

(4) *If* A *is a local domain then, under mild conditions on* A *(the class of rings allowed should include local rings of a finitely generated algebra over a complete local ring or over* $\mathbb{Z}$*), and* $f_1, \ldots, f_d$ *is a system of parameters for* A, *then for* $1 \le i \le d-1$, $(f_1, \ldots, f_i)A :_A f_{i+1} \subseteq \big((f_1, \ldots, f_i)A\big)^*$.

(5) If R is regular, then $I^* = I$ for every ideal I of R.

(6) For every module-finite extension ring R of S and every ideal I of R, $IS \cap R \subseteq I^*$.

These are all properties of tight closure in prime characteristic $p > 0$, and also of the theory of tight closure for Noetherian rings containing $\mathbb{Q}$ that we described in the Lecture of March 12. In characteristic $p > 0$, (4) holds for homomorphic images of Cohen-Macaulay rings, and for excellent local rings. If $R \supseteq \mathbb{Q}$, (4) holds if R is excellent. We will prove that (4) holds in prime characteristic for homomorphic images of Cohen-Macaulay rings quite soon. We have proved (5) in prime characteristic $p > 0$ for polynomial rings over a field, but not yet for all regular rings. To give the proof for all regular rings we need to prove that the Frobenius endomorphism is flat for all such rings, and we shall eventually use the structure theory of complete local rings to do this.

An extremely important open question is whether there exists a closure theory satisfying (1) — (6) for Noetherian rings that need not contain a field.

The final Theorem of the Lecture of March 12 makes it natural to consider the following variant notion of closure. Let R be any integral domain. Let R^+ denote integral closure of R in an algebraic closure $\overline{\mathcal{K}}$ of its fraction field $\mathcal{K}$. We refer to this ring as the *absolute integral closure* of R. R^+ is unique up to non-unique isomorphism, just as the algebraic closure of a field is. Any module-finite (or integral) extension domain S of R has fraction field algebraic over $\mathcal{K}$, and so S embeds in $\overline{\mathcal{K}}$. It follows that S embeds in R^+, since the elements of S are integral over R. Thus, R^+ contains an R-subalgebra isomorphic to any other integral extension domain of R: it is a maximal extension domain with respect to the property of being integral over R. R^+ is the directed union of its finitely generated subrings, which are module-finite over R. R^+ is also charactized as follows: it is a domain that is an integral extension of R, and every monic polynomial with coefficients in R^+ factors into monic linear polynomials over R^+.

Given an ideal $I \subseteq R$, the following two conditions on $f \in R$ are equivalent:

(1) $f \in IR^+ \cap R$.

(2) For some module-finite extension S of R, $f \in IS \cap R$.

The set of such elements, which is $IR^+ \cap R$, is denoted I^+, and is called the *plus closure* of I. (The definition can be extended to modules $N \subseteq M$ by defining N^+_M to be the kernel of the map $M \to R^+ \otimes_R (M/N)$.)

By the last Theorem of the Lecture Notes of March 12, which is property (6) above in characteristic $p > 0$, we have that

$$I \subseteq I^+ \subseteq I^*$$

in prime characteristic $p > 0$. Whether $I^+ = I^*$ in general under mild conditions for Noetherian rings of prime characterisitc $p > 0$ is another very important open question. It is not known to be true even in finitely generated algebras of Krull dimension 2 over a field.

However, there are some substantial positive results. It is known that under the mild conditions on the local domain R (e.g., when R is excellent), if I is generated by part of a system of parameters for R, then $I^+ = I^*$. See [K. E. Smith, *Tight closure of parameter ideals*, Inventiones Math. **115** (1994) 41–60]. Moreover, H. Brenner [H. Brenner, *Tight closure and plus closure in dimension two*, Amer. J. Math. **128** (2006) 531–539] proved that if R is the homogeneous coordinate ring of a smooth projective curve over the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime integer $p > 0$, then $I^* = I^+$ for homogeneous ideals primary to the homogeneous maximal ideal. In [G. Dietz, *Closure operations in positive* *characteristic and big Cohen-Macaulay algebras*, Thesis, Univ. of Michigan, 2005] the condition that the ideal be homogeneous is removed: in fact, there is a corresponding result for modules $N \subseteq M$ when M/N has finite length. Brenner's methods involve the theory of semi-stable vector bundles over a smooth curve (in fact, one needs the notion of a *strongly* semi-stable vector bundle, where "strongly" means that the bundle remains semi-stable after pullback by the Frobenius map).

One reason for the great interest in whether plus closure commutes with tight closure is that it is known that plus closure commutes with localization. Hence, if $I^* = I^+$ in general (under mild conditions on the ring) one gets the result that tight closure commutes with localization.

The notion of plus closure is of almost no help in understanding tight closure when the ring contains the rationals. The reason for this is the result established on pp. 4–5 of the Lecture Notes of March 12, which we restate formally here.

Theorem. *Let R be a normal Noetherian domain with fraction field $\mathcal{K}$ and let S be a module-finite extension domain with fraction field $\mathcal{L}$. Let $h = [\mathcal{L} : \mathcal{K}]$. If $\mathbb{Q} \subseteq R$, or, more generally, if h has an inverse in R, then $\dfrac{1}{h}\mathrm{Tr}_{\mathcal{L}/\mathcal{K}}$ gives an R-module retraction $S \to R$.* $\square$

It follows that if $\mathbb{Q} \subseteq R$ and R is a normal domain, then $I^+ = I$ for every ideal I of R. Many normal rings (in some sense most normal rings) that are essentially of finite type over $\mathbb{Q}$ are not Cohen-Macaulay, and so contain parameter ideals that are not tightly closed. This shows that plus closure is not a greatly useful notion in Noetherian domains that contain $\mathbb{Q}$.

Weakly F-regular rings and F-regular rings

We define a Noetherian ring R of prime characteristic $p > 0$ to be *weakly F-regular* if every ideal is equal to its tight closure, i.e., every ideal is tightly closed. We define R to be *F-regular* if all of its localizations are weakly F-regular. It is not known whether weakly F-regular implies F-regular, even for domains finitely generated over a field. This would follow if tight closure were known to commute with localization.

We have already proved that polynomial rings over a field of positive characteristic are weakly F-regular, and we shall prove that every regular ring of positive characteristic is F-regular. This is one reason for the terminology. The "F" suggest the involvement of the Frobenius endomorphism.

We shall soon show that a weakly F-regular ring is normal, and, if it is a homomorphic image of a Cohen-Macaulay ring, is itself Cohen-Macaulay.

Theorem. *A direct summand A of a weakly F-regular domain is weakly F-regular, and a direct summand of an F-regular domain is F-regular.*

Proof. Assume that R is weakly F-regular. If $f \in I_A^*$, then $f \in (IR)^* \cap A = IR \cap A = I$. Since the direct summand condition is preserved by localization on A, it follows that a direct summand of an F-regular domain is F-regular. $\square$

Examples of F-regular rings. Fix a field K of characteristic $p > 0$. Normal rings finitely generated over K by monomials are direct summand of regular rings, and so are F-regular. If X is an $r \times s$ matrix of indeterminates over K with $1 \le t \le r \le s$, then it is known that $K[X]/I_t(X)$ is F-regular, and that the ring generated by the $r \times r$ minors of X over K is F-regular (this is the homogeneous coordinate ring of the Grassmann variety). See [M. Hochster and C. Huneke, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. of Algeraic Geometry **3** (1994) 599–670], Theorem (7.14). We have already observed that these rings are direct summands of polynomial rings when K has characteristic 0, but this is not true in any obvious way when the characteristic is positive.

Splitting from module-finite extension rings

It is natural to attempt to characterize the Noetherian domains R such that R is a direct summand, as an R-module, of every module-finite extension ring S. We define a Noetherian ring R with this property to be a *splinter*. We then have the following result, which was actually proved in the preceding lecture, although it was not made explicit there.

Theorem. *Let R be a Noetherian domain.*

(a) *If R is a splinter, then every ideal of R is contracted from every integral extension.*

(b) *If R is a splinter, then R is normal.*

(c) *R is a splinter if and only if it is a direct summand of every module-finite domain extension.*

(d) *If $\mathbb{Q} \subseteq R$, then R is a splinter if and only if R is normal.*

Proof. For part (a), suppose $f, f_1, \ldots, f_h \in R$ and $f = \sum_{i=1}^h f_i s_i$ with the s_i in S. Then we have the same situation when S is replaced by $R[s_1, \ldots, s_h]$. Hence, it suffices to show that every ideal of R is contracted from every module-finite extension S. But then we have an R-linear retraction $\phi : S \to R$, and the result is part (a) of the Lemma at the top of p. 2 of the Lecture of February 16.

Part (b) has already been established in the fourth paragraph on p. 4 of the Lecture of March 12.

For part (c), we have already observed that S has a minimal prime $\mathfrak{p}$ disjoint from $R - \{0\}$, and it suffices to split the injection $R \hookrightarrow S/\mathfrak{p}$.

Finally, for part (d), the existence of the required splitting when S is a domain is proved at the bottom of p. 4 and top of p. 5 of the Lecture Notes of March 12, using field trace, and restated on p. 3 here. $\square$

The example on p. 5 of the Lecture Notes of March 12 shows that in positive characteristic p, a normal domain need not be a splinter. The property of being a splinter in characteristic p is closely related to the property of being weakly F-regular.

We first note the following fact: we shall not give the proof in these lectures, but refer the reader to [M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) 25–43] and [M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488].

Theorem. *Let R be a normal Noetherian domain. Then R is a direct summand of a module-finite extension of S if and only if every ideal of R is contracted from S.*

Of course, we know the "only if" part.

Corollary. *Let R be a normal Noetherian domain of positive characteristic p. Then R is a splinter if and only if for every ideal $I \subseteq R$, $I = I^+$.*

Corollary. *If R is a normal Noetherian domain and R is weakly F-regular, then R is a splinter.*

Proof. This is immediate from the preceding result, since $I^+ \subseteq I^*$. $\square$

We shall see quite soon that if R is weakly F-regular it is automatic that R is normal. If plus closure is the same as tight closure, then it would follow that R is weakly F-regular if and only if R is a splinter. This is an open question.

We have already observed that in characteristic $p > 0$, regular rings are weakly F-regular, although we have not prove this. Assuming this for the moment we have:

Corollary. *A regular ring that contains a field is a direct summand of every module-finite extension ring.*

This was conjectured by the author in 1969, and has been open question for regular rings that do not contain a field, such as polynomial rings over the integers, for 37 years. The case of dimension 3 was recently settled affirmatively in [R. C. Heitmann, *The direct summand conjecture in dimension three*, Annals of Math. (2) **156** (2002) 695–712]. The case of dimension 4 remains open for regular rings that do not contain a field.

It is also a major open question whether there exists a tight closure theory satisfying conditions (0) — (6) of p. 1 for Noetherian rings that need not contain a field. The existence of such a theory would imply that direct summands of regular rings are Cohen-Macaulay in general, and that regular rings are direct summands of all of their module-finite extensions in general. Such a theory would also settle many other open questions.