Math 615: Lecture of March 16, 2007

We next want to study weakly F-rings, i.e., Noetherian rings of prime characteristic $p > 0$ such that every ideal is tightly closed. Until further notice, all given rings R are assumed to be Noetherian, of prime characteristic $p > 0$.

Proposition. *The tight closure of the* (0) *ideal in* R *is the ideal of all nilpotent elements. Hence, if* $(0) = (0)^*$*, the* R *is reduced. In particular, every weakly F-regular ring is reduced.*

Proof. If u is nilpotent then $1 \cdot u^q = 0$ for all $q \gg 0$. Conversely, if $c \in R^\circ$ and $cu^q = 0$ for all $q \gg 0$, then for every minimal prime $\mathfrak{p}$ we have that $cu^q \in \mathfrak{p}$ for some q. Since $c \notin \mathfrak{p}$, we have that $u^q \in \mathfrak{p}$ and so $u \in \mathfrak{p}$. But the intersection of the minimal primes is the set of nilpotent elements of R, and so u is nilpotent. The remaining statements are now obvious. $\square$

Proposition. *Suppose that* $R = S \times T$ *is a product ring, with* $S, T \neq 0$*. Then for every ideal* $I \times J$ *of* $S \times T$*, where* $I \subseteq S$ *and* $J \subseteq T$ *are ideals,* $(I \times J)^*_R = I^*_S \times J^*_T$*.*

Proof. The first point is that $(S \times T)^\circ = (S^\circ) \times (T^\circ)$. Hence if $cs^q \in I^{[q]}$ for all $q \gg 0$ and $dt^q \in J^{[q]}$ for all $q \gg 0$, we have that

$$(c, d)(s, t)^q \in I^{[q]} \times J^{[q]} = (I \times J)^{[q]}$$

for all $q \gg 0$. The converse is also immediate. $\square$

Corollary. *A finite product* $R_1 \times \cdots \times R_h$ *is weakly F-regular if and only if every factor is weakly F-regular.* $\square$

Theorem. *If every principal ideal of* R *is tightly closed, then* R *is a product of normal domains.*

Proof. The fact that $(0) = (0)^*$ implies that R is reduced. We first show that R is a product of domains. If there are two or more minimal primes, the minimal primes can be partitioned into two nonempty sets. Call the intersection of one set I and the intersection of the other set J. Then $I \cap J = 0$, and $I + J$ is not contained in any minimal prime $\mathfrak{p}$, for otherwise, $\mathfrak{p}$ would have to contain both a minimal prime of I and a minimal prime of J, and would be equal to both of these. Hence we can choose $f \in I$ and $g \in J$ such that $f + g$ is not in any minimal prime of R, and so is a nonzerodivisor. Note that $fg \in I \cap J$, and so $fg = 0$. Now

$$(f + g)f^q = f^{q+1} = f(f + g)^q$$

for all q, so that $f \in (f + g)^* = (f + g)R$. Thus, we can choose $r \in R$ such that $f = r(f + g) = rf + rg$, and the $f - rf = rg$. Since $f \in I$ and $g \in J$, both sides must vanish, and so $f = rf$ and $rg = 0$. Now $r(f + g) = rf = f$, and

$$r^2(f + g) = r(rf + rg) = r(f + 0) = rf = f,$$

so that

$$(f+g)(r^2-r)=0.$$

Since $f+g$ is not a zerodivisor, we have that $r^2-r=0$. Since $rf=f$ is not 0 (or $f+g$ would be in the minimal primes containing g) $r \neq 0$. Since $rg=0$, $r \neq 1$. Therefore, R contains a non-trivial idempotent, and is a product of two rings. Both have the property that principal ideals are tightly closed, because a principal ideal of $S \times T$ is the product of a principal ideal of S and a principal ideal of T, and we may apply the Proposition above.

We may apply this argument repeatedly and so write R as a finite product of rings with the property that every principal ideal is tightly closed, and such that none of the factors is a product. Each of the factors must have just one minimal prime, and so is a domain. It remains to see that if principal ideals are tightly closed in a domain R, then R is normal. Suppose that $f,\, g \in R$, $g \neq 0$, and f/g is integral over R. Let $S = R[f/g]$, which is module-finite over R. Then $f = g(f/g) \in gS$, and so $f \in (gR)^*$. But $(gR)^* = gR$, and so $f \in gR$, i.e., $f/g \in R$, as required. □

We next want to show that, under mild conditions on R, if R is weakly F-regular then R is Cohen-Macaulay. Before giving the proof, we make some comments about Cohen-Macaulay rings in general.

Cohen-Macaulay rings

In this section, we assume that given rings are Noetherian, but make no assumption about the characteristic. In particular, given rings need not contain a field.

We have defined the notion of a Cohen-Macaulay ring in the case of a finitely generated $\mathbb{N}$-graded K-algebra R with $R_0 = K$. We have also defined the notion of a Cohen-Macaulay local ring, and define a Noetherian ring to be Cohen-Macaulay if all of its local rings are Cohen-Macaulay. We first note:

Lemma. *Let $(R,\, m,\, K)$ be a local ring and let I be an ideal of height h in R. Then there is a sequence of elements $x_1,\, \ldots\,, x_h$ in I that is part of a system of parameters for R.*

Proof. If $h=0$ we may take the empty sequence. If $h \geq 1$, then I is not contained in the union of the minimal primes of R, or else we would have that I is contained in one of them and has height 0. Choose $x_1 \in I$ not in any minimal prime of R. Then x_1 is part of a system of parameters. We use induction. Suppose that $x_1,\, \ldots\,, x_i \in I$ have been chosen so that they are part of a system of paramters with $i < h$. The minimal primes of $(x_1,\, \ldots\,, x_i)R$ all have height $\leq i < h$, and so I is not contained in any of them and also not contained in their union. Choose $x_{i+1} \in I$ not in any minimal prime of $(x_1,\, \ldots\,, x_i)R$. Then $x_1,\, \ldots\,, x_{i+1}$ is also part of a system of parameters. □

Corollary. *If $(R,\, m)$ is Cohen-Macaulay and P is a prime ideal of R, the R_P is Cohen-Macaulay.*

Proof. Suppose that $h = \text{height}\,(P) = \dim\,(R_P)$. Choose $x_1, \ldots, x_h \in P$ part of a system of parameters for R. Then $x_1, \ldots, x_h$ is a regular sequence in R, and, hence, also in R_P, by flatness. $\square$

Theorem. *Let R be a Noetherian ring. The following conditions are equivalent:*

(1) *R is Cohen-Macaulay, i.e., R_P is Cohen-Macaulay for every prime ideal P.*

(2) *R_m is Cohen-Macaulay for every maximal ideal m.*

(3) *For every proper ideal I of R,* $\text{depth}_I R = \text{height}\,(I)$.

Proof. (1) $\Rightarrow$ (2) is obvious, while (2) $\Rightarrow$ (1) because each R_P is a localization of R_m for some maximal ideal containing P. Now assume (2) and suppose that I has height h. Choose a maximal regular sequence $x_1, \ldots, x_d$ in I on R. Then $R/(x_1, \ldots, x_d)R$ has depth 0 on $I/(x_1, \ldots, x_d)R$, and this remains true after we localize at an associated prime P of $R/(x_1, \ldots, x_d)R$ that contains $I(x_1, \ldots, x_d)R$. Hence, $x_1, \ldots, x_d$ is also a maximal regular sequence in R_P, which shows that $d = h$, since R_P is Cohen-Macaulay of dimension h. Thus, (2) $\Rightarrow$ (3).

Finally, assume (3). Let P be any prime ideal of R of height h. Then P contains a regular sequence of length $\text{height}\,(P) = \dim\,(R_P)$, and this sequence remains a regular sequence when we localize at P. Hence, (3) $\Rightarrow$ (1). $\square$

Theorem. *If R is Cohen-Macaulay, so is the polynomial ring in n variables over R.*

Proof. By induction, we may assume that $n = 1$. Let $\mathcal{M}$ be a maximal ideal of $R[X]$ lying over m in R. We may replace R by R_m and so we may assume that (R, m, K) is local. Then $\mathcal{M}$, which is a maximal ideal of $R[x]$ lying over m, corresponds to a maximal ideal ideal of $K[x]$: each of these is generated by a monic irreducible polynomial f, which lifts to a monic polynomial F in $R[x]$. Thus, we may assume that $\mathcal{M} = mR[x] + FR[X]$. Let $x_1, \ldots, x_d$ be a system of parameters in R, which is also a regular sequence. We may kill the ideal generated by these elements, which also form a regular sequence in $R[X]_{\mathcal{M}}$. We are now in the case where R is an Artin local ring. It is clear that the height of $\mathcal{M}$ is one. Because F is monic, it is not a zerodivisor: a monic polynomial over any ring is not a zerodivisor. This shows that the depth of $\mathcal{M}$ is one, as needed. $\square$

Theorem. *If R is a finitely generated graded K-algebra with $[R]_0 = K$, then R is Cohen-Macaulay in the graded sense if and only if R is Cohen-Macaulay.*

Proof. Let m be the homogeneous maximal ideal. If R_m is Cohen-Macaulay, choose a maximal regular sequence in m consisting of homogeneous elements (necessarily of positive degree), say $F_1, \ldots, F_h$. When we kill these elements, we know that in $R/(F_1, \ldots, F_h)R$, the homogeneous elements of the ideal $m/(F_1, \ldots, F_h)R$ are all contained in the union of the associated primes of $R/(F_1, \ldots, F_h)R$. By the Proposition on homogeneous prime avoidance from the bottom of p. 4 of the Lecture of January 26, $m/(f_1, \ldots, f_h)R$ itself is contained in one of these associated primes, and so $m/(f_1, \ldots, f_hR)$ is an associated prime. This is preserved when we localize at $m/(f_1, \dots, f_h)R$, and so R_m has depth 0 once we kill $(f_1, \dots, f_h)R_m$. Therefore, $f_1, \dots, f_h$ is a maximal regular sequence in R_m, and this implies that $h = \dim(R_m) = \dim(R)$. Thus, R is Cohen-Macaulay in the graded sense.

Now suppose that R is Cohen-Macaulay in the graded sense. Then R is a module-finite extension of a polynomial ring $A = K[X_1, \dots, X_n]$, and the polynomial ring is Cohen-Macaulay. Any maximal ideal $\mathfrak{q}$ of R lies over a maximal ideal $\mathfrak{n}$ of A. These have the same height, since we have both the going up and going down theorems in this situation: A is normal, and R is A-free and, hence, torsion-free over A. Since R is A-free, a regular sequence in $A_{\mathfrak{n}}$ is regular on $R_{\mathfrak{n}}$, which is free and, hence, faithfully flat over A, and will remain regular on $R_{\mathfrak{q}}$, which is a localization of $R_{\mathfrak{n}}$. $\square$

We next observe:

Theorem. *Let (R, m, K) be a local ring and $M \neq 0$ a finitely generated R-module of depth s on m. Then every nonzero submodule N of M has dimension at least s.*

Proof. We use induction on s. If $s = 0$ there is nothing to prove. Assume $s > 0$ and that the result holds for smaller s. If M has a submodule $N \neq 0$ of dimension $\leq s-1$, we may choose N maximal with respect to this property. If N' is any nonzero submodule of M of dimension $< s$, then $N' \subseteq N$. To see this, note that $N \oplus N'$ has dimension $< s$, and maps onto $N + N' \subseteq M$, which therefore also has dimension $< s$. By the maximality of N, we must have $N + N' = N$. Since $\mathrm{depth}_m M \geq 1$, we can choose $x \in m$ not a zerodivisor on M, and, hence, also not a zerodivisor on N. We claim that x is not a zerodivisor on $\overline{M} = M/N$, for if $u \in M - N$ and $xu \in N$, then $Rxu \subseteq N$ has dimension $< s$. But this module is isomorphic with $Ru \subseteq M$, since x is not a zerodivisor, and so $\dim(Ru) < s$. But then $Ru \subseteq N$. Consequently, multiplication by x induces an isomorphism of the exact sequence $0 \to N \to M \to \overline{M} \to 0$ with the sequence $0 \to xN \to xM \to x\overline{M} \to 0$, and so this sequence is also exact. But we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & xN & \longrightarrow & xM & \longrightarrow & x\overline{M} & \longrightarrow & 0 \end{array}$$

where the vertical arrows are inclusions. By the nine lemma, or by an elementary diagram chase, the sequence of cokernels $0 \to N/xN \to M/xM \to \overline{M}/x\overline{M} \to 0$ is exact. Since x is a nonzerodivisor on N and M, $\dim(N/xN) = \dim(N) - 1 < s-1$, while $\mathrm{depth}_m M/xM = s-1$. This contradicts the induction hypothesis. $\square$

Corollary. *If (R, m, K) is a Cohen-Macaulay local ring, then for every minimal prime $\mathfrak{p}$ of R, $\dim(R/\mathfrak{p}) = \dim(R)$.*

Proof. If $\mathfrak{p}$ is minimal, then $\mathfrak{p} \in \mathrm{Ass}(R)$ and so $R/\mathfrak{p} \hookrightarrow R$. By the preceding Theorem, $\dim(R/\mathfrak{p}) \geq \mathrm{depth}_m R = \dim(R)$, while the other inclusion is obvious. $\square$

Thus, a Cohen-Macaulay local ring cannot exhibit the kind of behavior one observes in $R = K[[x,\,y,\,z]]/\bigl((x,\,y)\cap(z)\bigr)$: this ring has two minimal primes. One of them, $\mathfrak{p}_1$, generated by the images of x and y, is such that $R/\mathfrak{p}_1$ has dimension 1. The other, $\mathfrak{p}_2$, generated by the image of z, is such that $R/\mathfrak{p}_2$ has dimension 2.

A Noetherian ring is called *catenary* if for any two prime ideals $P \subseteq Q$, any two saturated chains of primes joining P to Q have the same length. If R is catenary, then so is R/I for every ideal I, since primes containing I are in bijective correspondence with primes of R containing I, and saturated chains of primes in R/I joining P/I to Q/I, where $I \subseteq P \subseteq Q$ and P, Q are primes of R, correspond to saturated chains of primes of R joinig P to Q. Similarly, any localization of a catenary ring is catenary. M. Nagata gave the first examples of Noetherian rings that are not catenary: there is a local domain $(R,\,m,\,K)$ of dimension 3, for example, containing saturated chains $0 \subset Q \subset m$ and $0 \subset P_1 \subset P_2 \subset m$, where all inclusions are strict. See [M. Nagata, *Local rings*, Interscience, New York, 1962], Appendx A1, pp. 204–205. Although Q has height one and $\dim(R) = 3$, the dimension of R/Q is 1. Nagata also showed that even when a Noetherian ring is catenary, the polynomial ring in one variable over it need not be.

A Noetherian ring R is called *universally catenary* if every finitely generated R-algebra is catenary. Cohen-Macaulay rings are universally catenary, as we show in the two results below.

Theorem. *A Cohen-Macaulay ring R is catenary, and for any two prime ideals $P \subseteq Q$ in R, every saturated chain of prime ideals joining P to Q has length* $\mathrm{height}\,(Q) - \mathrm{height}\,(P)$. *Hence, every finitely generated algebra over a Cohen-Macaulay ring is catenary.*

Proof. The issues are unaffected by localizing at Q. Thus, we may assume that R is local and that Q is the maximal ideal. There is part of a system of parameters of length $h = \mathrm{height}\,(P)$ contained in P, call it $x_1,\,\ldots\,,x_h$, by the Lemma at the beginning of this section. This sequence is a regular sequence on R and in so on R_P, which implies that its image in R_P is system of parameters. We now replace R by $R/(x_1,\,\ldots\,,x_h)$. Both the dimension and depth of R have decreased by h, so that R is still Cohen-Macaulay. Q and P are replaced by their images, which have heights $\dim(R) - h$ and 0, and $\dim(R) - h = \dim(R/(x_1,\,\ldots\,,x_h)$. We have therefore reduced to the case where R is local and P is a minimal prime. We know that $\dim(R) = \dim(R/P)$, and so at least one saturated chain from P to Q has length $\mathrm{height}\,(Q) - \mathrm{height}\,(P) = \mathrm{height}\,(Q) - 0 = \dim(R)$. To complete the proof, it will suffice to show that all saturated chains from P to Q have the same length, and we may use induction on $\dim(R)$. Copnsider two such chains, and let their smallest elements other than P be P_1 and P_1'. Choose an element x in P_1 not in any minimal prime, and an element y of P_1' not in any minimal prime. Then xy is a nonzerodivisor in R, and P_1, P_1' are both minimal primes of xy. The ring $R/(xy)$ is Cohen-Macaulay of dimension $\dim(R) - 1$. The result now follows from the induction hypothesis applied to $R/(xy)$: the images of the two saturated chains (omitting P from each) give saturated chains joining $P_1/(xy)$ (respectively, $P_1'/(xy)$) to $Q/(xy)$ in $R/(xy)$. These have the same length, and, hence, so did the original two chains. $\square$

Corollary. *Cohen-Macaulay rings are universally catenary, i.e., a finitely generated algebra over a Cohen-Macaulay ring is catenary.*

Proof. Such an algebra is a homomorphic image of a polynomial ring in finitely many variables over a Cohen-Macaulay ring, which is again Cohen-Macaulay, and homomorphic images of catenary rings are catenary. □