Math 615: Lecture of March 19, 2007

Colon-capturing in homomorphic images of Cohen-Macaulay rings

We will need the following two preliminary results:

Lemma (prime avoidance for cosets). *Let S be any commutative ring, $x \in S$, $I \subseteq S$ an ideal and $P_1, \ldots, P_k$ prime ideals of S. Suppose that the coset $x + I$ is contained in $\bigcup_{i=1}^k P_i$. Then there exists j such that $Sx + I \subseteq P_j$.*

Proof. If $k = 1$ the result is clear. Choose $k \geq 2$ minimum giving a counterexample. Then no two P_i are comparable, and $x + I$ is not contained in the union of any $k - 1$ of the P_i. Now $x = x + 0 \in x + I$, and so x is in at least one of the P_j: say $x \in P_k$. If $I \subseteq P_k$, then $Sx + I \subseteq P_k$ and we are done. If not, choose $i_0 \in I - P_k$. We can also choose $i \in I$ such that $x + i \notin \bigcup_{j=1}^{k-1} P_i$. Choose $u_j \in P_j - P_k$ for $j < k$, and let u be the product of the u_j. Then $ui_0 \in I - P_k$, but is in P_j for $j < k$. It follows that $x + (i + ui_0) \in x + I$, but is not in any P_j, $1 \leq j \leq k$, a contradiction. $\square$

Lemma. *Let S be a Cohen-Macaulay local ring, let P be a prime ideal of S of height h, and let $x_1, \ldots, x_{i+1}$ be part of a system of parameters of $R = S/P$. Let $y_1, \ldots, y_h \in P$ be part of a system of parameters for S (we have such a sequence by the first Lemma of the preceding section on Cohen-Macaulay rings). Then there exist elements $\widetilde{x}_1, \ldots, \widetilde{x}_{i+1}$ of S such that $\widetilde{x}_j$ maps to x_j modulo P, $1 \leq j \leq i+1$, and $y_1, \ldots, y_h, \widetilde{x}_1, \ldots, \widetilde{x}_{i+1}$ is part of a system of parameters for S.*

Proof. We construct the $\widetilde{x}_j$ recursively. Suppose that the $\widetilde{x}_j$ for $j < k+1 \leq i+1$ have been chosen so that $y_1, \ldots, y_h, \widetilde{x}_1, \ldots, \widetilde{x}_k$ is part of a system of parameters for S. Here, k is allowed to be 0 (i.e., we may be choosing $\widetilde{x}_1$). We want to choose an element of $x_{k+1} + P$ that is not in any minimal prime of $y_1, \ldots, y_h, \widetilde{x}_1, \ldots, \widetilde{x}_k$, and these all have height at most $h + k$. By the Lemma on prime avoidance for cosets, if $\widetilde{x}_{k+1} + P$ is contained in the union, then $Sx_{k+1} + P$ is contained in one of them, say Q. Working modulo P we have that Q/P is a minimal prime $x_1, \ldots, x_{k+1}$ of height at most $h + k - h = k$. This is a contradiction, since $x_1, \ldots, x_{k+1}$ is part of a system of parameters in S/P, and so any minimal prime must have height at least $k + 1$. $\square$

Theorem (colon-capturing). *Let (R, m, K) be a local domain of prime characteristic $p > 0$, and suppose that R is a homomorphic image of a Cohen-Macaulay ring of characteristic p. Let $x_1, \ldots, x_{i+1}$ by part of a system of parameters in R. Then*

$$(x_1, \ldots, x_i) :_R x_{i+1} \subseteq (x_1, \ldots, x_i)^*.$$

Proof. Suppose that $R = S/P$, where S is Cohen-Macaulay of characteristic p, and let Q be the inverse image of m in S. Then R is also a homomorphic image of S_Q, since $S_Q/PS_Q \cong (S/P)_Q = R_Q = R_m = R$. Hence, we may assume that S is local. Choose $y_1, \ldots, y_h$ and $\widetilde{x}_1, \ldots, \widetilde{x}_{i+1}$ as in the preceding Lemma. Since P is a minimal prime of $(y_1, \ldots, y_h)$ in S, we can choose $\widetilde{c} \in S - P$ and an integer $N > 0$ such that $\widetilde{c}P^N \in (y_1, \ldots, y_h)S$. Let $c \neq 0$ be the image of $\widetilde{c}$ in R. Suppose that $fx_{i+1} = f_1x_1 + \cdots + f_ix_i$ in R. Then we can choose elements $\widetilde{f}$ and $\widetilde{f}_1, \ldots, \widetilde{f}_i$ in S that lift f and $f_1, \ldots, f_i$ respectively to S. This yields an equation

$$\widetilde{f}\widetilde{x}_{i+1} = \widetilde{f}_1\widetilde{x}_1 + \cdots + \widetilde{f}_i\widetilde{x}_i + \Delta$$

in S, where $\Delta \in P$. Then for all $p^e = q \geq N$ we have

$$\widetilde{f}^q\widetilde{x}_{i+1}^q = \widetilde{f}_1^{\,q}\widetilde{x}_1^q + \cdots + \widetilde{f}_i^q\widetilde{x}_i^q + \Delta^q$$

We may multiply both sides by $\widetilde{c}$, and use the fact that $\widetilde{c}\Delta^q \in cP^N \subseteq (y_1, \ldots, y_h)$ to conclude that

$$(*) \quad \widetilde{c}\widetilde{f}^q\widetilde{x}_{i+1}^q \in (\widetilde{x}_1^q, \ldots, \widetilde{x}_i^q, y_1, \ldots, y_h)S$$

But $y_1, \ldots, y_h, \widetilde{x}_1^q, \ldots, \widetilde{x}_{i+1}^q$ is a permutable regular sequence in S, and so $(*)$ implies that

$$\widetilde{c}\widetilde{f}^q \in (\widetilde{x}_1^q, \ldots, \widetilde{x}_i^q, y_1, \ldots, y_h)S.$$

When we consider this modulo P, We have that $(y_1, \ldots, y_h)$ is killed, and so

$$cf^q \in (x_1^q, \ldots, x_i^q)$$

for all $q \geq N$, and this gives the desired conclusion. $\square$

Weak F-regularity: localization at maximal ideals and the Cohen-Macaulay property

We next want to prove that the property of being weakly F-regular is local on the maximal ideals of R. From this we will deduce that a weakly F-regular ring that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay. We need two preliminary results.

Lemma. *Let R be any Noetherian ring, let M be a finitely generated R-module and $N \subseteq M$ a submodule. Then N is the intersection of a (usually infinite) family of submodules Q of M such that every M/Q is killed by a power of a maximal ideal of R.*

In particular, every ideal I of R is an intersection of ideals that are primary to a maximal ideal of R.

Proof. Let $u \in M - N$. Consider the family of submodules $M_1 \subseteq M$ such that $N \subseteq M$ and $u \notin M_1$. This family is nonempty, since it contains N. Therefore it has a maximal element Q. It will suffice to show that M/Q is killed by a power of a maximal ideal of R. Note that every nonzero submodule of M/Q contains the image of u, or else its inverse image in M will strictly contain Q but will not contain u.

We may replace M by M/Q and u by its image in M/Q. It therefore suffices to show that if $u \neq 0$ is in every nonzero submodule of M, then M is killed by a power of a maximal ideal, which is equivalent to the assertion that $\mathrm{Ass}\,(M)$ consists of a single maximal ideal. Let $P \in \mathrm{Ass}\,(M)$ and suppose that $P = \mathrm{Ann}_R v$, where $v \neq 0$ is in M. Then $Rv \cong R/P$, and every nonzero element has annihilator P. But $u \in Rv$, and so $P = \mathrm{Ann}_R u$. It follows that every associated prime of M is the same as $\mathrm{Ann}_R u$, and so there is only one associated prime. It remains to show that P is maximal. Suppose not, and consider $R/P \hookrightarrow M$. It will suffice to show that there is no element in all the nonzero ideals of R/P. Thus, it suffices to show that if $S = R/P$ is a Noetherian domain of dimension at least one, there is no nonzero element in all the nonzero ideals. This is true, in fact, even if we localize at a nonzero prime ideal m of S, for in S_m, there is no element in all of the ideals $m^n S_m$. $\square$

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let $\mathfrak{A}$ be an ideal of R.*

(a) *If $\theta : R \to S$ is such that S is flat Noetherian R-algebra and, in particular, if S is a localization of R, then $\theta(\mathfrak{A}^*_R) \subseteq (\mathfrak{A}S)^*_S$.*

(b) *Let m be a maximal ideal of R and suppose that $\mathfrak{A}$ is an m-primary ideal. Let $f \in R$. Then $f \in \mathfrak{A}^*_R$ if and only if $f/1 \in (\mathfrak{A}R_m)^*_{R_m}$.*

(c) *Under the hypotheses of part (b), $\mathfrak{A}$ is tightly closed in R if and only if $\mathfrak{A}R_m$ is tightly closed in R_m.*

Proof. (a) Let $f \in \mathfrak{A}^*_R$. The equation $cf^q \in \mathfrak{A}^{[q]}$ implies $\theta(c)\theta(f)^q \in (\mathfrak{A}S)^{[q]}$, and so we need only see that if $c \in R^\circ$ then $c \in S^\circ$. Suppose, to the contrary, that c is in a minimal prime $\mathfrak{q}$ of S. It suffices to see that the contraction $\mathfrak{p}$ of $\mathfrak{q}$ to R is minimal. But $R_\mathfrak{p} \to S_\mathfrak{q}$ is still faithfully flat, and the maximal ideal of $S_\mathfrak{q}$ is nilpotent, which implies that $\mathfrak{p}R_\mathfrak{p}$ is nilpotent, and so $\mathfrak{p}$ is minimal.

For part (b), we see from (a) that if $f \in \mathfrak{A}^*$ then $f \in (\mathfrak{A}R_m)^*$. We need to prove the converse. Suppose that $c_1 \in R_m^\circ$ has the property that $cf_1^q \in \mathfrak{A}^{[q]}R_m = (\mathfrak{A}R_m)^{[q]}$ for all $q \gg 0$. Then c_1 has the form c/w where $c \in R$ and $w \in R - m$. We may replace c_1 by wc_1, since w is a unit, and therefore assume that $c_1 = c/1$ is the image of $c \in R$. We next want to replace c by an element with the same image in R_m that is not in any minimal prime of R. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal primes of R that are contained in m, so that the ideals $\mathfrak{p}_j R_m$ for $1 \leq j \leq k$ are *all* of the minimal primes of R_m. It follows that the image of $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ is nilpotent in R_m, and so we can choose an integer $N > 0$ such that $I = (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k)^N$ has image 0 in R_m. If $c + I$ is contained in the union of the minimal primes of R, then by the coset form of prime avoidance, it follows that $cR + I \subseteq \mathfrak{p}$ for some minimal prime $\mathfrak{p}$ of R. Since $I \subseteq \mathfrak{p}$, we have that $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k \subseteq \mathfrak{p}$, and it follows that $\mathfrak{p}_j = \mathfrak{p}$ for some j, where $1 \leq j \leq k$. But then $c \in \mathfrak{p}_j$, a contradiction, since $c/1$ is not in any minimal prime of R°. Hence, we can choose $f \in I$ such that $c + f \in R^\circ$, and $c + f$

also maps to $c/1$ in R. We change notation and assume $c \in R^\circ$. Then $cf^q/1 \in \mathfrak{A}^{[q]}R_m$ for all $q \gg 0$. Since $\mathfrak{A}^{[q]}$ is primary to m, the ring $R/\mathfrak{A}^{[q]}$ has only one maximal ideal, $m/\mathfrak{A}^{[q]}$, and is already local. Hence,

$$R/\mathfrak{A}^{[q]} \cong (R/fA^{[q]})_m = R_m/\mathfrak{A}^{[q]}R_m.$$

It follows that $cf^q \in \mathfrak{A}^{[q]}$ for all $q \gg 0$, and so $f \in \mathfrak{A}^*_R$, as required.

Part (c) is immediate from part (b) and the observation above that $R_m/\mathfrak{A}R_m = R/\mathfrak{A}$, so that any element of $R_m/\mathfrak{A}R_m$ is represented by an element of R. $\square$

Remark. Part (a) holds for any map $R \to S$ of Noetherian rings of prime characteristic $p > 0$ such that R° maps into S°. We have already seen another example, namely when $R \hookrightarrow S$ are domains.

Theorem. *The following conditions on R are equivalent.*

(1) *R is weakly F-regular.*

(2) *Every ideal of R primary to a maximal ideal of R is tightly closed.*

(3) *For every maximal ideal m of R, R_m is weakly F-regular.*

Proof. Statements (2) and (3) are equivalent by part (c) of the preceding Proposition, and (1) $\Rightarrow$ (2) is clear. Assume (2), and let I be any ideal of R. We need only show that I is tightly closed. If not, let $f \in I^* - I$. Since I is the intersection of the ideals containing I that are primary to maximal ideals, there is an ideal $\mathfrak{A}$ of R primary to a maximal ideal m such that $I \subseteq \mathfrak{A}$ and $f \notin \mathfrak{A}$. Since $\mathfrak{A}$ is tightly closed and $I \subseteq \mathfrak{A}$, we have $I^* \subseteq \mathfrak{A}$, and so $f \in \mathfrak{A}$, a contradiction. $\square$