Math 615: Lecture of March 28, 2007

Consider a complete local ring $(R,\, m,\, K)$. If K has characteristic 0, then $\mathbb{Z} \to R \to K$ is injective, and $\mathbb{Z} \subseteq R$. Moreover, no element of $W = \mathbb{Z} - \{0\}$ is in m, since no element of W maps to 0 in $R/m = K$, and so every element of $\mathbb{Z} - \{0\}$ has an inverse in R. By the universal mapping property of localization, we have a unique map of $W^{-1}\mathbb{Z} = \mathbb{Q}$ into R, and so R is an equicharacteristic 0 ring. We already know that R has a coefficient field. We also know this when R has prime characteristic $p > 0$, i.e., when $\mathbb{Z}/p\mathbb{Z} \subseteq R$.

We now want to develop the structure theory of complete local rings when R need not contain a field. From the remarks above, we only need to consider the case where K has prime characteristic $p > 0$, and we shall assume this in the further development of the theory. The coefficient rings that we are about to describe also exist in the complete separated quasi-local case, but, for simplicity, we only treat the Noetherian case.

We shall say that V is a *coefficient ring* if it is a field or if it is complete local of the form $(V,\, pV,\, K)$, where K has characteristic $p > 0$. If R is complete local we shall say that V is a *coefficient ring for R* if V is a coefficient ring, $V \subseteq R$ is local, and the induced map of residue fields is an isomorphism. We shall prove that coefficient rings always exist.

In the case where the characteristic of K is $p > 0$, there are three possibilities. It may be that $p = 0$ in R (and V), in which case V is a field: we have already handled this case. It may be that p is not nilpotent in V: in this case it turns out that V is a Noetherian discrete valuation domain (DVR), like the p-adic integers. Finally, it may turn out that p is not zero, but is nilpotent.

We are aiming to prove the following two results. Like the other theorems we have been proving about the structure of complete local rings, they are due to I. S. Cohen.

Theorem. *Let $(R,\, m,\, K)$ be a complete local ring of mixed characteristic. Then R has a coefficient ring.*

Theorem. *Let $(W,\, pW,\, K)$ be a coefficient ring of mixed charateristic such that p is nilpotent. Then W has the form $V/p^h V$, where $(V,\, pV,\, K)$ is a coefficient ring that is a complete Noetherian discrete valuation ring.*

Before proving these two results, which will take a considerable effort, we want to give several consequences.

Theorem. *Let R be a complete local ring of mixed characteristic.*

(a) *R is a homomorphic image of a power series ring $V[[X_1,\, \ldots,\, X_n]]$ over a complete Noetherian discrete valuation ring $(V,\, pV,\, K)$, where n is the embedding dimension of R/pR.*

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(b) *If R is a domain, or more generally, if p is part of a system of parameters for R, then R is module-finite over a formal power series ring $V[[x_2, \ldots, x_{d-1}]]$, where $d = \dim(R)$ and V is a complete Noetherian discrete valuation ring that is a coefficient ring for R.*

(c) *Suppose that R is regular of Krull dimension d and that V is a complete Noetherian discrete valuation ring that is a coefficient ring for R. If $p \notin m^2$, then $R \cong V[[x_2, \ldots, x_d]]$, a formal power series ring. If R is regular and $p \in m^2$, then $R \cong V[[x_1, \ldots, x_d]]/(f)$, where the numerator is a formal power series ring and $f = p - g$ with g is in the square of the maximal ideal of $V[[x_1, \ldots, x_d]]$.*

Proof. (a) Let W be a coefficient ring for R and let V be a coefficient ring that is a discrete valuation ring that maps onto W. Choose $f_1, \ldots, f_n \in R$ that map onto a minimal set of generators of the maximal ideal of R/pR. Then p together with the $f_1, \ldots, f_n$ map onto generators of m. By part (a) of the Proposition stated at the top of p. 4 of the Lecture Notes of March 23, there is a map $W[[x_1, \ldots, x_n]] \to R$ that takes $x_1, \ldots, x_n$ to $f_1, \ldots, f_n$ respectively, and this map is a surjection by part (c) of that same Proposition, with $A = W$ and $\mu = pW$. Hence, we have surjections

$$V[[x_1, \ldots, x_n]] \twoheadrightarrow W[[x_1, \ldots, x_n]] \twoheadrightarrow R,$$

as required.

(b) Since p is part of a system of parameters, it is not nilpotent, and a coefficient ring (V, p, K) for R must be a Noetherian discrete valuation ring. Let $f_2, \ldots, f_d \in m$ be elements that extend p to a system of parameters for R. By parts (a) and (b) of the Proposition cited above, we have a map $V[[x_2, \ldots, x_d]] \to R$ such that R is module finite over the image. Since the $\dim(R) = d$, the image has dimension d, and since $V[[x_2, \ldots, x_d]]$ is a domain of dimension d, the map cannot have a kernel.

(c) If R is regular and $p \notin m^2$, then we can extend p to a minimal set of generators $p, f_2, \ldots, f_d$ of m, and we have a map $V[[x_2, \ldots, x_d]] \to R$ that is injective and such that R is module-finite over the image by part (b). But we are also in the situation of part (a), so that this map is surjective, and this gives the required isomorphism of R with a formal power series ring.

Now suppose that $p \in m^2$. We proceed as in part (a), but choose $f_1, \ldots, f_d$ so that they are a minimal set of generators of m. Let $T = V[[x_1, \ldots, x_d]]$, the formal power series ring, and let m_T be its maximal ideal. Then we have a surjection $T \twoheadrightarrow R$. Since $p \in m^2$, the kernel of this map must contain an element of the form $p - g$, where $g \in m_T^2$. But $f = p - g \in m_T - m_T^2$, and so $T/(f)$ is a regular local ring of dimension d that maps onto R. Since $T/(f)$ is regular, it is a domain, and it follows that the map $T/(f) \twoheadrightarrow R$ cannot have a non-trivial kernel. Thus, $T/(f) \cong R$, as required. $\square$

A regular local ring of mixed characteristic $p > 0$ is called *unramified* if $p \notin m^2$ and *ramified* if $p \in m^2$.

Example. Let $R = V[[x]]/(px)$, where (V, pV, K) is a coefficient ring, and x is a power series indeterminate over V. The image of V in R is isomorphic with V and is a coefficient ring. R is one-dimensional, and is not module-finite over a regular ring: cf. problem 5. of Problem Set #5.

It remains to prove the results of I. S. Cohen about coefficient rings for complete local rings of mixed characteristic, including the statement that they exist. The following elementary fact is critical in carrying this through.

Lemma. *Let (R, m, K) be local with K of prime characteristic $p > 0$. If $r, s \in R$ are such that $r \equiv s \bmod m$, and $n \geq 1$ is an integer, then for all $N \geq n - 1$, with $q = p^N$ we have that $r^q \cong s^q \bmod m^n$.*

Proof. This is clear if $n = 1$. We use induction. If $n > 1$, we know from the induction hypothesis that $r^q \equiv y^q \bmod m^N$ if $N \geq n - 2$, and it suffices to show that $r^{pq} \equiv y^{pq} \bmod m^{N+1}$. Since $r^q = s^q + u$ with $u \in m^N$, we have that $r^{pq} = (s^q + u)^p = s^{pq} + puw + u^p$, where puw is a sum of terms from the binomial expansion each of which has the form $\binom{pq}{j} s^j u^{p-j}$ for some j, $1 \leq j \leq p - 1$, and in each of these terms the binomial coefficient is divisible by p. Since $u \in m^N$ and $p \cdot 1_R \in m$, $puw \in m^{N+1}$, while $u^p \in m^{Np} \subseteq m^{N+1}$ as well. $\square$