Math 615: Lecture of April 13, 2007

Note that given a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

of a finitely generated R-module M and an additive map L we have that

$$L(M) = L(M_n/M_{n-1}) + L(M_{n-1}),$$

and, by induction on n, that

$$L(M) = \sum_{j=1}^{n} L(M_j/M_{j-1}).$$

In particular, $[M] \in G_0(R)$ is

$$\sum_{j=1}^{n} [M_j/M_{j-1}].$$

The following result gives a presentation of the Grothendieck group.

Theorem. *Let R be a Noetherian ring. $G_0(R)$ is generated by the elements $[R/P]$, as P runs through all prime ideals of R. If P is prime and $x \in R - P$, then $[R/(P + xR)] = 0$, and so if $R/Q_1, \ldots, R/Q_k$ are all the factors in a prime filtration of $[R/(P + xR)]$, we have that $[R/Q_1] + \cdots + [R/Q_k] = 0$. The relations of this type are sufficient to generate all relations on the classes of the prime cyclic modules.*

Proof. The first statement follows from the fact that every finitely generated module over a Noetherian ring R has a finite filtration in which the factors are prime cyclic modules. The fact that $[R/(P + xR)] = 0$ follows from the short exact sequence

$$0 \to R/P \xrightarrow{x} R/P \to R/(P + xR) \to 0,$$

which implies $[R/P] = [R/P] + [R/(P + xR)]$ and so $[R/(P + xR)] = 0$ follows.

Now, for every $M \in \mathcal{M}$, fix a prime cyclic filtration of M. We need to see that if we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

that the relation $[M] = [M'] + [M'']$ is deducible from ones of the specified type. We know that M' will be equal to the sum of the classes of the prime cyclic modules occurring in its chosen prime filtration, and so will M''. These two prime cyclic filtrations together induce a prime cyclic filtration $\mathcal{F}$ of M, so that the information $[M] = [M'] + [M'']$ is conveyed by setting $[M]$ equal to the sum of the classes of the prime cyclic modules in these specified filtrations of $[M]$ and $[M']$. But $\mathcal{F}$ will not typically be the specified filtration of $[M]$, and so we need to set the sum of the prime cyclic modules in the specified filtration of M equal to the sum of all those occurring in the specified filtrations of M' and M''.

Thus, we get sufficiently many relations to span all relations if for all finitely generated modules M and for all pairs of possibly distinct prime cyclic filtrations of M, we set the sum of the classes of the prime cyclic modules coming from one filtration equal to the corresponding sum for the other. But any two filtrations have a common refinement. Take a common refinement, and refine it further until it is a prime cyclic filtration again. Thus, we get sufficiently many relations to span if for every finitely generated module M and for every pair consisting of a prime cyclic filtration of M and a refinement of it, we set the sum of the classes coming from one filtration to the sum of those in the other. Any two prime cyclic filtrations may then be compared by comparing each to a prime cyclic filtration that refines them both.

In refining a given prime cyclic filtration, each factor R/P is refined. Therefore, we get sufficiently many relations to span if for every R/P and every prime cyclic filtration of R/P, we set $[R/P]$ equal to the sum of the classes in the prime cyclic filtration of R/P. Since $\mathrm{Ass}\,(R/P) = P$, the first submodule of a prime cyclic filtration of R/P will be isomorphic with R/P, and will therefore have the form $x(R/P)$, where $x \in R - P$. If the other factors are $R/Q_1, \ldots, R/Q_k$, then these are the factors of a filtration of $(R/P)/x(R/P) = R/(P + xR)$. Since $[x(R/P)] = [R/P]$, the relation we get is

$$[R/P] = [R/P] + [R/Q_1] + \cdots + [R/Q_k],$$

which is equivalent to

$$[R/Q_1] + \cdots + [R/Q_k] = 0,$$

and so the specified relations suffice to span all relations. $\square$

We can immediately deduce as a consequence the theorem of Murthy stated in the Lecture Notes of April 6.

Theorem (M. P. Murthy). *If R is a normal domain, then $\mathcal{C}\ell\,(R) \cong [G_0(R)]_1$ in such a way that the generator of $\mathcal{C}\ell\,(R)$ corresponding to a height one prime P is mapped to the image of R/P in $[G_0(R)]_1$.*

Proof. We know that $G_0(R)$ is the free group on the classes of the R/P, P prime, modulo relations obtained from prime cyclic filtrations of $R/(P + xR)$, $x \notin P$. We shall show that if we kill all the $[R/Q]$ for Q of height 2 or more, all relations are also killed except those coming from $P = (0)$, and the image of any relation corresponding to a prime cyclic filtration of R/xR corresponds precisely to $\mathrm{div}\,(x)$. Clearly, if $P \neq 0$ and $x \notin P$, any prime containing $P + xR$ strictly contains P and so has height two or more. Thus, we need only consider relations on the R/P for P of height one coming from prime cyclic filtrations of R/xR, $x \neq 0$. Clearly, R does not occur, since R/xR is a torsion module, and occurrences of R/Q for Q of height ≥ 2 do not matter. We need only show that for every prime P of height one, the number of occurrences of R/P in any prime cyclic filtration of R/xR is exactly k, where $P^{(k)}$ is the P-primary component of xR. But we can do this calculation after localizing at P: note that all factors corresponding to other primes become 0, since some element in the other prime not in P is inverted. Then $xR_P = P^k R_p$, and we need to show that any prime cyclic filtration of R_P/xR_P has k copies of R_P/PR_P, where we know that $xR_P = P^k R_P$. Notice that $(R_P,\, PR_P)$ is a DVR, say (V, tV), and $xR_P = t^k V$. The number of nonzero factors in any prime cyclic filtration of V/t^kV is the length of V/t^kV over V, which is k, as required: the only prime cyclic filtration without repetitions is

$$0 \subset t^{k-1}V \subset t^{k-2}V \subset \cdots \subset t^2V \subset tV \subset V. \qquad \square$$

We next restate and then prove the local form of the Hilbert syzygy theorem stated in the Lecture Notes of April 6. The result is entirely analogous to the corresponding result in the graded case treated in the second problem of Problem Set #3.

Theorem (Hilbert syzygy theorem for regular local rings). *Let (R, m, K) be a regular local ring of Krull dimension n, and let M be a finitely generated nonzero R-module. Then M is free if and only if* $\mathrm{depth}(M) = n$. *If M is not free and M_1 is any first module of syzygies of M,* $\mathrm{depth}(M_1) = \mathrm{depth}(M) + 1$. *Hence, M has a finite free resolution by finitely generated free modules, and any shortest such free resolution of M has length* $n - \mathrm{depth}(M)$.

Proof. For the first statement we use induction on $\dim(R)$. If $n = 0$ then R is a field, every module is free, and there is nothing to prove. Assume that $n > 0$. It is clear that if M is a nonzero free module then its depth is n. Suppose that M has depth n. In particular, $\mathrm{depth}(M) \geq 1$ and we can choose $x \in m$ not in m^2 nor in any minimal prime of M. Then M/xM has depth $n-1$ over R/xR, which is again regular. Thus, M/xM is free by the induction hypothesis: let $u_1, \ldots, u_h$ be elements of M whose images are a free basis for M/xM. These elements span M by Nakayama's Lemma. To complete the proof of this part, it suffices to show that they have no nonzero relation over R. Let N denote the module of all relations on $u_1, \ldots, u_h$ over R. If $(f_1, \ldots, f_h) \in N$ is a relation, so that $f_1u_1 + \cdots f_hu_h = 0$, then we may consider this relation modulo xR. Since the images of the u_j are a free basis for M/xM, it follows that every f_j is in xR, and can be written xg_j for some $g_j \in R$. Then $x(g_1u_1 + \cdots g_hu_h) = 0$, and since x is not a zerodivisor on M, we have that $g_1u_1 + \cdots g_hu_h = 0$. Thus $(f_1, \ldots, f_h) = x(g_1, \ldots, g_h)$ with $(g_1, \ldots, g_h) \in N$, and we consequently have that $N = xN$. By Nakayama's Lemma, $N = 0$, and it follows that M is free on the basis $u_1, \ldots, u_h$.

The remaining statements now follow from part (a) of the second problem of Problem Set #2 exactly as in the graded case. $\square$

We have now completed the proof of unique factorization in regular local rings, following M. P. Murthy.

We want to note another proof of a variant of the Hilbert syzygy theorem for finitely generated modules over polynomial rings, based on Gröbner basis ideas. The argument is based on Schreyer's method for computing modules of relations or syzygies, which is described beginning near the bottom of p. 2 of the Lecture of January 24, and continuing on pp. 3, 4, and 5. We review the method, which is very simple.

Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K, and let $M \subseteq F$ be a submodule of F, where F is free with ordered basis $b_1, \ldots, b_s$. Let $g_1, \ldots, g_r$ be a Gröbner basis for M. (We shall momentarily impose a mild condition on the ordering of the g_i.) We may view the relations on $g_1, \ldots, g_r$ as a submodule R^r, for which we denote the standard basis as $e_1, \ldots, e_r$. Schreyer's method asserts that the module of all relations on $g_1, \ldots, g_r$ is generated by certain standard relations as follows. Suppose that $i < j$ and that $\mathrm{in}(g_i) = \mu_i b_k$, $\mathrm{in}(g_j) = \mu_j b_k$ involve the same element b_k of $b_1, \ldots, b_s$. Then we can write

$$(*_{ij}) \qquad \frac{\mu_j}{\mathrm{GCD}(\mu_i,\, \mu_j)} g_i - \frac{\mu_i}{\mathrm{GCD}(\mu_i,\, \mu_j)} g_j = \sum_{t=1}^{r} q_{ijt} g_t$$

where the left hand side is a standard expression for division of the left hand side by $g_1, \ldots, g_r$. The remainder is 0 in each case by the Buchberger criterion. The displayed equation implies that

$$(\#_{ij}) \qquad \frac{\mu_j}{\mathrm{GCD}(\mu_i,\, \mu_j)} e_i - \frac{\mu_i}{\mathrm{GCD}(\mu_i,\, \mu_j)} e_j - \sum_{t=1}^{r} q_{ijt} e_t$$

is a relation on $g_1, \ldots, g_r$. This is a typical standard relation, and we saw that these not only generate the module of all relations, but are, in fact, a Gröbner basis for it with respect to a suitable monomial order on R^r. Moreover, the initial term of $(\#_{ij})$ is

$$(\dagger_{ij}) \qquad \frac{\mu_j}{\mathrm{GCD}(\mu_i,\, \mu_j)} e_i.$$

We now make an almost trivial observation:

Lemma. *Let hypotheses and notations be as above and suppose that $g_1, \ldots, g_r$ have been ordered so that if $i > g$ and $\mathrm{in}(g_i) = \mu_i b_k$ and $\mathrm{in}(g_j) = \mu_j b_k$ involve the same element b_k of the ordered basis $b_1, \ldots, b_s$ for F then $\mu_i > \mu_j$ in lexicographic order on the monomials of R. (This does not depend on what the monomial order on F is: one can always order the g_i so that this condition is satsified.) Suppose that the initial terms of the g_i involve only the x_i for $i \geq h$. Then the initial terms of the standard relations on $g_1, \ldots, g_r$ involve only the variables x_i for $i \geq h+1$.*

Proof. Since only the variables $x_h, \ldots, x_n$ occur and $\mu_i > \mu_j$ in lexicographic order, we must have that the highest power of x_h occurring in μ_i is at least that occurring in μ_j: call the latter x_h^a. It follows that x_h^a is also the highest power of x_h occurring in $\mathrm{GCD}(\mu_i,\,\mu_j)$, and so x_h does not occur in the initial term shown in $(\dagger_{ij})$ of the standard relation $(\#_{ij})$. $\square$

Given any finitely generated module M over R its first module of syzygies M_1 is a submodule of a free module. Even if all the variables occur in generators of the initial module for M_1, after at most n repetitions of Schreyer's method, each time with the Gröbner basis obtained ordered as indicated in the Lemma above, one obtains a Gröbner basis for the module of syzygies such that every initial term is simply one of the e_j. We can now complete our variant proof of the HIlbert syzygy theorem by showing that a module with a Gröbner basis of this form is free.