Math 615: Lecture of April 16, 2007

We next note the following fact:

Proposition. *Let R be any ring and $F = R^n$ a free module. If $f_1, \dots, f_n \in F$ generate F, then $f_1, \dots, f_n$ is a free basis for F.*

Proof. We have a surjection $R^n \twoheadrightarrow F$ that maps $e_i \in R^n$ to f_i. Call the kernel N. Since F is free, the map splits, and we have $R^n \cong F \oplus N$. Then N is a homomorphic image of R^n, and so is finitely generated. If $N \neq 0$, we may preserve this while localizing at a suitable maximal ideal m of R. We may therefore assume that (R, m, K) is quasilocal. Now apply $K \otimes_R \_$. We find that $K^n \cong K^n \oplus N/mN$. Thus, $N = mN$, and so $N = 0$. $\square$

The final step in our variant proof of the Hilbert syzygy theorem is the following:

Lemma. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, let F be a free R-module with ordered free basis $e_1, \dots, e_s$, and fix any monomial order on F. Let $M \subseteq F$ be such that* $\mathrm{in}(M)$ *is generated by a subset of $e_1, \dots, e_s$, i.e., such that M has a Gröbner basis whose initial terms are a subset of $e_1, \dots, e_s$. Then M and F/M are R-free.*

Proof. Let S be the subset of $e_1, \dots, e_s$ generating $\mathrm{in}(M)$, and suppose that S has r elements. Let $T = \{e_1, \dots, e_s\} - S$, which has $s - r$ elements. Let $G \cong R^{s-r}$ be the free submodule of F spanned by T. By the Theorem on the bottom of p. 2 of the Lecture Notes of January 12, the images of the monomials not in $\mathrm{in}(M)$ are a K-vector space basis for F/M. These monomials, which are simply those involving an element of T, are obviously also a K-vector space basis for G. It follows that the composite R-linear map $G \subseteq F \twoheadrightarrow F/M$ is an isomorphism of K-vector spaces. Since it is R-linear, it is also an isomorphism of R-modules. It is clear that $M + G = F$, since $\mathrm{in}(M) \cup \mathrm{in}(G) = S \cup T = \mathrm{in}(F)$. Since no element of $G - \{0\}$ is killed in F/M, the sum is direct, i.e., $F = M \oplus G$. Let $g_1, \dots, g_r$ be elements of a Gröbner basis for M whose initial terms are the elements of S. Then $g_1, \dots, g_r$ together with T are s elements that generate $F \cong R^s$, and so they form a free basis for R^s by the preceding Proposition. It follows that $g_1, \dots, g_r$ is a free basis for M. $\square$

We have now proved a "global" result on modules of syzygies over a polynomial ring: every finitely generated module has an n th module of syzygies that is free. It follows that every finitely generated module has a finite resolution by finitely generated free modules. This means in turn that if $R = K[x_1, \dots, x_n]$, a polynomial ring over a field, then $G_0(R) = \mathbb{Z}\gamma$, where $\gamma = [R]$, and the universal additive map is given by torsion-free rank. It follows just as in the local case that $[G_0(R)]_1 = 0$, i.e., that $\mathcal{C}\ell(R) = 0$, which gives a new proof that a polynomial ring over a field is a UFD, quite different from the usual one.

We next want to discuss projective modules over a Noetherian ring. A module P over R is *projective* if for every surjective map $M \twoheadrightarrow N$ the map $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$

is surjective. It follows at once that $\mathrm{Hom}_R(P, \_)$ is a (covariant) exact functor from R-modules to R-modules. It is easy to see that free modules are projective: to lift a map $f : F \to N$ when F has free basis $\mathcal{B}$, for every $b \in \mathcal{B}$ one chooses $u_b \in M$ such that $u_b \mapsto f(b) \in N$, and one may then define $g : F \to M$ such that $g(b) = u_b$ for all $b \in \mathcal{B}$. The direct sum of two modules is projective if and only if both are projective: this follows from the fact that $\mathrm{Hom}_R(P \oplus Q, M)$ may be identified, functorially in M, with $\mathrm{Hom}_R(P, M) \oplus \mathrm{Hom}_R(Q, M)$. Hence, a direct summand of a free module is projective. Note that since free modules are flat, and since a direct sum of two modules is flat if and only if both are (because of the identification, functorial in M, of

$$(P \oplus Q) \otimes_R M \cong (P \otimes_R M) \oplus (Q \otimes_R M),)$$

it follows that projective modules are flat.

We have the following in great generality:

Proposition. *Let R be any ring and P an R-module. The following conditions are equivalent:*

(1) *P is projective.*

(2) *Every surjective map $f : M \twoheadrightarrow P$ splits, where M is an aribtrary R-module.*

(3) *P is a direct summand of a free module.*

Moreover, if P is finitely generated, then P is projective if and only if it is a direct summand of a finitely generated free module.

Proof. We have seen in the paragraph above that (3) $\Rightarrow$ (1). If P is projective and we have $f : M \twoheadrightarrow P$, the identity map $\mathbf{1}_P : P \to P$ lifts to a map $g : P \to M$: this means that $f \circ g = \mathbf{1}_P$, so that g is the required splitting. Finally, (2) $\Rightarrow$ (3) because if P satisfies (2) and we map a free module $F \twoheadrightarrow P$, the map splits, and so P is a direct summand of F. If P is finitely generated, we may take F to be finitely generated. $\square$

If P is a finitely generated projective module, we know that there exists a projective module Q such that $P \oplus Q$ is free. Q is called a *complement* for P. Q itself need not be free. If there exists a free module G such that $P \oplus G$ is a finitely generated free module, G is called a *free complement* for P.

Proposition. *Let R be any ring and let P be a projective module that has a finite resolution by finitely generated free modules. Then P has a free complement.*

Proof. We use induction on the length of the free resolution. If the resolution is

$$0 \to F_1 \to F_0 \to P \to 0$$

then the map $F_0 \twoheadrightarrow P$ splits, and $F_0 \cong P \oplus F_1$. Now suppose that the resolution has length $k > 1$. Let $Q = \mathrm{Ker}\,(F_0 \twoheadrightarrow P)$. Then $F_0 \cong P \oplus Q$, so that Q is projective, and Q

has a free resolution of length at most $k-1$. By the induction hypothesis, we can choose a finitely generated free module G such $Q \oplus G = H$ is a finitely generated free module. Then $P \oplus (Q \oplus G) = F_0 \oplus G$ is free, and since $Q \oplus G = H$, we have that H is a free complement for P. $\square$

Hence, given our results for polynomial rings, we have an easy proof of the following:

Theorem. *Let R be a polynomial ring $K[x_1, \ldots, x_n]$ over a field K. Then every finitely generated projective R-module has a free complement.* $\square$

In the mid 1950s Serre asked whether every finitely generated projective module over a polynomial ring is free. This was not answered until 1976, when D. Quillen and A. Suslin gave proofs indepedently. Another, simpler, proof was found soon thereafter by Vaserstein. One way of attacking the problem is as follows.

One wants to prove that when R is a polynomial ring over a field, if $P \oplus R^k$ is free, then P is free. It suffices to show this when $k = 1$. For then, since $P' \oplus R$ is free, with $P' = P \oplus R^{k-1}$, one can conclude that P' is free, and the result follows by induction on k. Thus, once one knows that every finitely generated projective module has a free complement, showing that every finitely generated projective module is free is equivalent to showing that if $P \oplus R = R^n$ then P is free.

Over any ring R, giving a projective module P such that $P \oplus R = R^n$ is equivalent to giving a surjective map $R^n \twoheadrightarrow R$. The kernel of this map, call it P, then has the property that $P \oplus R = R^n$, for the map $R^n \twoheadrightarrow R$ is split, and so the short exact sequence

$$0 \to P \to R^n \to R \to 0$$

is split. Giving a surjective map $R^n \to R$ is the same as giving a $1 \times n$ matrix $\begin{pmatrix} g_1 & \ldots, & g_n \end{pmatrix}$ whose entries generate the unit ideal of R. This determines P. Note that we have elements $f_1, \ldots, f_n \in R$ such that $f_1 g_1 + \cdots + f_n g_n = 1$, and the map of $R \to R^n$ sending $1 \mapsto (f_1, \ldots, f_n)$ gives the splitting.

The projective module P, if free, must have rank $n-1$. In fact, P is free if and only if it is generated by $n-1$ elements $\rho_2, \ldots, \rho_n$. For in this case, these elements together with $\rho_1 = (f_1, \ldots, f_n)$ generate R^n, and so we have a surjetion $R^n \to R^n$ which is, necessarily, a bijection. If we make the ρ_i into the rows of a matrix, the condition that the rows generate R^n (equivalently, that the rows be a free basis for R^n) is that the matrix be invertible.

A row whose entries generate the unit ideal is called a *unimodular* row. The *unimodular row problem* asks the following: given a unimodular row over a ring R, can it be completed to a square matrix whose determinant is 1? This question has an affirmative answer over R for all size rows if and only if for every projective module P over R such that $P \oplus R$ is free of finite rank, P is free. If every finitely generated projective R-module has a free complement, an affirmative answer to the unimodular row problem implies that every finitely generated projective R-module is free.

As mentioned above, this is the case for polynomial rings in finitely many variables over a field. We want to give one example where there is a projective module with free complement of rank one that is not free. Let $R = \mathbb{R}[X, Y, Z]/(X^2+Y^2+Z^2) = \mathbb{R}[x, y, z]$, which may be thought of as the coordinate ring of the real 2-sphere S^2. Elements of R may be thought of as real-valued continuous functions on S^2. Note that $\bigl(x\ y\ z\bigr)$ is a unimodular row, since $x^2 + y^2 + z^2 = 1$ in R. This row cannot be completed to a 3×3 matrix whose determinant is 1 if the entries of the matrix are in R, nor even if the entries are allowed to be arbitrary continuous real-valued functions on S^2. It follows that

$$P = \mathrm{Ker}\,(R^3 \twoheadrightarrow R),$$

where the map has matrix $\bigl(x\ y\ z\bigr)$, is a projective module over R that is not free but such that $P \oplus R = R^3$. To show that we cannot complete the matrix, suppose that we can, and let the second row be $\bigl(f\ \ g\ \ h\bigr)$ where f, g, h are continuous real-valued functions on S_2. Since the determinant of the matrix is constantly equal to 1, for every point $(a, b, c) \in S^2$, if we substitute a, b, c for the variables the first two rows of the matrix are linearly independent. Thus, (a, b, c) and

$$w(a, b, c) = \bigl(f(a, b, c), g(a, b, c), h(a, b, c)\bigr)$$

are linearly independent, and since the vector (a, b, c) is normal to the tangent plane to the sphere at the point (a, b, c), the vector $w(a, b, c)$ has a nonzero projection $v(a, b, c)$ on the tangent plane to S^2 at (a, b, c) that varies continuously with (a, b, c). This constructs a continuous nonzero vector field on S^2, which contradicts a well-known theorem in toplogy ("you can't comb the hair on a billiard ball").

After tensoring with the complex numbers, one can complete the row. Working now over $\mathbb{C}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$, we see that the matrix

$$\begin{pmatrix} x & xi + y & z \\ 0 & -z & -xi + y \\ 1 & 0 & 0 \end{pmatrix}$$

has determinant 1: expand with respect to the third row. If we subtract i times the first column from the second, we get the matrix we want:

$$\begin{pmatrix} x & y & z \\ 0 & -z & -xi + y \\ 1 & -i & 0 \end{pmatrix}.$$

The flatness of the Frobenius endomorphism for regular rings

We shall return to the subject of projective modules, but we first want to establish the assertion made earlier that the Frobenius endomorphism is flat for every regular Noetherian ring of prime characteristic $p > 0$. To do so, we want to reduce to the case where the ring is complete local. We first observe the following:

Proposition. *Let $\theta : (R, m, K) \to (S, \mathfrak{n}, L)$ be a homomorphism of local rings that is* local*, i.e., $\theta(m) \subseteq \mathfrak{n}$. Then S is flat over R if and only if for every injective map $N \hookrightarrow M$ of finite length R-modules, $S \otimes_R N \hookrightarrow S \otimes_R M$ is injective.*

Proof. The condition is obviously necessary. We shall show that it is sufficient. Since tensor commutes with direct limits and every injection $N \hookrightarrow M$ is a direct limit of injections of finitely generated R-modules, it suffices to consider the case where $N \subseteq M$ are finitely generated. Suppose that some $u \in S \otimes_R N$ is such that $u \mapsto 0$ in $S \otimes_R M$. It will suffice to show that there is also such an example in which M and N have finite length. Fix any integer $t > 0$. Then we have an injection

$$N/(m^t M \cap N) \hookrightarrow M/m^t M$$

and there is a commutative diagram

$$\begin{array}{ccc} S \otimes_R N & \xrightarrow{\iota} & S \otimes_R M \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ S \otimes_R \big(N/(m^t M \cap N)\big) & \xrightarrow{\iota'} & S \otimes_R \big(M/m^t M\big) \end{array} .$$

The image $f(u)$ of u in $S \otimes_R \big((N/(m^t M \cap N)\big)$ maps to 0 under ι', by the commutativity of the diagram. Therefore, we have the required example provided that $f(u) \neq 0$. However, for all $h > 0$, we have from the Artin-Rees Lemma that for every sufficiently large integer t, $m^t M \cap N \subseteq m^h N$. Hence, the proof will be complete provided that we can show that the image of u is nonzero in

$$S \otimes_R (N/m^h N) \cong S \otimes_R \big((R/m^h) \otimes_R N\big) \cong (R/m^h) \otimes_R (S \otimes_R N) \cong (S \otimes_R N)/m^h (S \otimes_R N).$$

But

$$m^h (S \otimes_R N) \subseteq \mathfrak{n}^h (S \otimes_R N),$$

and the result follows from the fact that the finitely generated S-module $S \otimes_R N$ is $\mathfrak{n}$-adically separated. $\square$

Lemma. *Let $(R, m, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism of local rings. Then S is flat over R if and only if $\widehat{S}$ is flat over $\widehat{R}$, and this hold iff $\widehat{S}$ is flat over R.*

Proof. If S is flat over R then, since $\widehat{S}$ is flat over S, we have that $\widehat{S}$ is flat over R. Conversely, if $\widehat{S}$ is flat over R, then S is flat over R because $\widehat{S}$ is faithfully flat over S: if $N \subseteq M$ is flat but $S \otimes_R N \to S \otimes_R M$ has a nonzero kernel, the kernel remains nonzero when we apply $\widehat{S} \otimes_S \_$, and this has the same effect as applying $\widehat{S} \otimes_R \_$ to $N \subseteq M$, a contradiction.

We have shown that $R \to S$ is flat if and only $R \to \widehat{S}$ is flat. If $\widehat{R} \to \widehat{S}$ is flat then since $R \to \widehat{R}$ is flat, we have that $R \to \widehat{S}$ is flat, and we are done. It remains only to show that if $R \to S$ is flat, then $\widehat{R} \to \widehat{S}$ is flat. By the Proposition, it suffices to show that if $N \subseteq M$ have finite length, then $\widehat{S} \otimes N \to \widehat{S} \otimes M$ is injective. Suppose that both modules are killed by m^t. Since S/m^tS is flat over R/m^t, if Q is either M or N we have that

$$\widehat{S} \otimes_{\widehat{R}} Q \cong \widehat{S}/m^t\widehat{S} \otimes_{\widehat{R}/m^t\widehat{R}} Q \cong \widehat{S}/m^t\widehat{S} \otimes_{R/m^t} Q \cong \widehat{S} \otimes_R Q,$$

and the result now follow because $\widehat{S}$ is flat over R. □

We are now ready to prove:

Theorem. *Let R be a regular Noetherian ring of prime characteristic $p > 0$. Then the Frobenius endomorphism $F : R \to R$ is flat.*

Proof. To distinguish the two copies of R, we let S denote the right hand copy, so that $F : R \to S$. The issue of flatness is local on R, and if P is prime, then $(R - P)^{-1}S$ is the localization of S at the unique prime Q lying over P (if we remember that S is R, then Q is P), since the p th power of every element of $S - Q$ is in the image of $R - P$. Hence, there is no loss of generality in replacing R by R_P, and we henceforth assume that (R, m, K) is local. By the preceding Lemma, $F : R \to R$ is flat if and only if the induced map $\widehat{R} \to \widehat{R}$ is flat, and this map is easily checked to be the Frobenius endomorphism on $\widehat{R}$. We have now reduced to the case where R is a complete regular local ring. By the structure theory for complete local rings, we may assume without loss of generality that $R = K[[x_1, \dots, x_n]]$ where K is a field of characteristic p. By the Theorem on p. 2 of the Lecture Notes of February 19, the Frobenius endomorphism $F : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ makes $K[x_1, \dots, x_n]$ into a free algebra over itself. It follows that it is flat over itself, and this remains true when we localize at $(x_1, \dots, x_n)$. By the preceding Lemma, we still have flatness after we complete both rings. Completing yields

$$F : K[[x_1, \dots, x_n]] \to K[[x_1, \dots, x_n]],$$

which proves the flatness result we need. □

We can now give the application of this result that we have been intending for some time.

Theorem. *Let R be a regular Noetherian ring of prime characteristic $p > 0$. Then every ideal I of R is tightly closed.*

Proof. Suppose $u \in I^* - I$ and $c \in R$ is not in any minimal prime and satisfies $cu^q \in I^{[q]}$ for all $q \gg 0$. We may replace R by its localization at a maximal ideal in the support of $(I + Ru)/I$, I by its expansion to the local ring, and u by its image in the local ring. The image of c in this local ring is still not in any minimal prime, i.e., it is not 0. We still have that $u \in I^* - I$. Thus, we may assume without loss of generality that R is local. Then for some q_0,

$$c \in \bigcap_{q \geq q_0} I^{[q]} :_R u^q = \bigcap_{q \geq q_0} (I :_R u)^{[q]} \subseteq \bigcap_{q \geq q_0} m^{[q]} \subseteq \bigcap_{q \geq q_0} m^q = (0),$$

a contradiction. Note that the fact that $I^{[q]} :_R u^q = (I :_R u)^{[q]}$ used in this argument is a consequence of the flatness of the Frobenius endomorphism. $\square$

Projective modules over Noetherian rings

We now return to the subject of projective modules. For finitely generated projective modules over Noetherian rings there are some interesting characterizations.

Theorem. *Let P be a finitely presented module over a quasilocal ring (R, m, K) (in particular, it suffices if R is local and P is finitely generated). Then the following conditions are equivalent:*

(1) *P is free.*

(2) *P is projective.*

(3) *P is flat.*

(4) *The map $m \otimes_R P \to P$ sending $u \otimes v \mapsto uv$ is injective (and so gives an isomorphism $m \otimes_R P \cong mP$).*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), while (3) $\Rightarrow$ (4) follows by applying $\_ \otimes_R P$ to the injection $m \hookrightarrow R$. It remains to prove the difficult implication (4) $\Rightarrow$ (1).

Choose a minimal set of generators $u_1, \ldots, u_n$ for M and map R^n onto P such that $(r_1, \ldots, r_n)$ is sent to $r_1u_1 + \cdots + r_nu_n$. Let N be the kernel of the surjection $R^n \twoheadrightarrow P$, so that we have a short exact sequence $0 \to N \to R^n \to P \to 0$. We also have a short exact sequence $0 \to m \to R \to K \to 0$: think of this as written vertically with m at the top and K at the bottom. Then we may tensor the two sequences together to get the following array (all tensor products are taken over R):

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & m \otimes N \arrow[r] \arrow[d] & m \otimes R^n \arrow[r] \arrow[d] & m \otimes P \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & K \otimes N \arrow[r, "f"] \arrow[d] & K \otimes R^n \arrow[r, "g"] \arrow[d] & K \otimes P \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows are obtained by applying $m \otimes \_$, $R \otimes \_$, and $K \otimes \_$, respectively to the short exact sequence $0 \to N \to R^n \to P \to 0$, and the columns are obtained by applying $\_ \otimes N$,

$\_ \otimes R^n$, and $\_ \otimes P$, respectively, to the short exact sequence $0 \to m \to R \to K \to 0$. The exactness of the rows and columns shown follows from the right exactness of tensor, with two exceptions: the injective arrow on the left in the middle row comes from the fact that tensoring with R simply yields the short exact sequence with which we started, while the injectivity of α is the hypothesis in (4). (We also have an injection at the top of the middle column because R^n is free, but we don't need this.)

It is easy to see that the four squares in the diagram commute.

The minimality of the set of generators $u_1, \ldots, u_n$ implies that g is an isomorphism of K^n with K^n, and the fact that M is finitely presented implies that N is finitely generated. To complete the proof it suffices to show that $K \otimes N = 0$, for then, by Nakayama's lemma, we have that $N = 0$. But if $N = 0$ then $R^n \to M$ is an isomorphism. To show that $K \otimes N$ is 0, it suffices to prove that the map f is injective.

Suppose that u is an element in the kernel of f. Choose $v \in N$ that maps to u. The image of v in R^n (we still call it v) maps to 0 in $K \otimes R^n$: we can go around the square on the lower left the other way, and u is killed by f. It follows that v is the image of an element w in $m \otimes R^n$. Suppose that w maps to x in $m \otimes M$. Then $\alpha(x) = 0$, because we can go around the square on the upper right the other way, and the image of v in M must be 0 because $v \in N$. But α is injective! Therefore, $x = 0$, which shows that w is the image of an element y in $m \otimes N$. Since w maps to v, y maps to v in N (the map $N \to R^n$ is injective), and this implies that v maps to 0 in $K \otimes N$. But v maps to u, and so $u = 0$. We are done: we have shown that f is injective! $\square$