Math 615, Fall 2007 **Problem Set #1**
Due: Friday, February 2

1. Let $\gamma > 1$ be an irrational real number. Consider the monomial order $>_\gamma$ on $K[x_1, x_2]$ defined by letting $W_\gamma(x_1^{a_1}x_2^{a_2}) = \gamma a_1 + a_2$ and defining $\mu' >_\gamma \mu$ precisely if $W(\mu') > W(\mu)$. Show that if $\gamma \neq \gamma'$, then $>_\gamma$ is different from $>_{\gamma'}$. Thus, there are at least $2^{\aleph_0}$ monomial orders on $K[x_1, x_2]$.

2. Let $\gamma_1 > \gamma_2 > \cdots > \gamma_n = 1$ be $n \geq 2$ positive real numbers linearly independent over the rational numbers $\mathbb{Q}$. Let $W(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = \sum_{j=1}^n \gamma_i a_i$, and define a monomial order on $K[x_1, \ldots, x_n]$ by $\mu' > \mu$ precisely if $W(\mu') > W(\mu)$.

(a) Show that for any monomial $\mu \neq 1$ and any monomial μ', there exists an integer $k > 0$ such that $\mu^k > \mu'$. Explain why this shows that $>$ cannot coincide with lexicographic order.

(b) Show that there there are monomials μ and μ' such that $\deg(\mu') < \deg(\mu)$ but $\mu' > \mu$. (This shows that $>$ cannot coincide with homogeneous lexicographic order, nor with reverse lexicographic order.)

3. Let $R = K[x_1, x_2, x_3]$, a polynomial ring over the field K. with the monomial order given by reverse lexicographic order.

(a) Let $g_1 = x_1x_2 + x_3^2$ and $g_2 = x_2^2 + x_1x_3$. Use the Buchberger algorithm to determine a Gröbner basis for $I = (g_1, g_2)R$.

(b) Let $h_1 = x_1^2 + x_2x_3$ and $h_2 = x_2^2 + x_1x_3$. Use the Buchberger algorithm to determine a Gröbner basis for $J = (h_1, h_2)R$.

[Note: the K-automorphism of R that sends x_1, x_2, x_3 to x_3, x_2, x_1, respectively, carries g_1 and g_2 to h_1 and h_2. Thus, the ideals in parts (a) and (b) are "the same" modulo renumbering of the variables.]

4. Let F be free R-module with ordered basis $e_1, \ldots, e_s$ over a polynomial ring $R = K[x_1, \ldots, x_n]$, where K is a field. Let $g_1, \ldots, g_r$ be elements of $F - \{0\}$. Suppose that for two indices $i \neq j$, g_i and g_j have the property that *all* of their terms involve the same element e_t of the basis (this is automatic if $F = R$), and that their initial terms $c_i\mu_ie_t$ and $c_j\mu_je_t$, where $c_i, c_j \in K - \{0\}$ and μ_i, μ_j are monomials in R, are relatively prime, i.e., $\mathrm{GCD}(\mu_i, \mu_j) = 1$. Prove that there is a standard expression for $G_{ij} = c_j\mu_jg_i - c_i\mu_ig_j$ for division with respect to $g_1, \ldots, g_r$ such that the remainder h_{ij} is 0.

5. Let $R = K[x_1, \ldots, x_N]$, where $N = \binom{n+1}{2} + n + 1 = \binom{n+2}{2}$, and define f_k for $1 \leq k \leq n$ as follows. Let $m = \binom{k+1}{2}$ and let $f_k = (x_m + x_{m+k+1})x_{m+1}x_{m+2}\cdots x_{m+k}$. For example, $f_1 = (x_1 + x_3)x_2$, $f_2 = (x_3 + x_6)x_4x_5$, $f_3 = (x_6 + x_{10})x_7x_8x_9$, etc. Let $I = (f_1, \ldots, f_n)R$. Using hlex, find generators for $\mathrm{in}(I)$ and find a reduced Gröbner basis for I.

6. Let $R = K[x_1, \ldots, x_{2n}]$ be the polynomial ring in $2n$ variables over K. Let P be the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \end{pmatrix}.$$

Determine $\mathrm{in}(P)$ and a Gröbner basis for P using homogeneous lexicographic order.