

Throughout,  $K$  is a field, and  $K \subseteq L$  is a field extension.

1. Let  $K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$ . Prove that if  $\text{in}(I)$  is generated by square-free monomials, i.e., is a radical ideal, then  $I$  is a radical ideal. Deduce that the ideal described in Problem 6. of the first problem set is radical.
2. Let  $\Sigma$  be a finite simplicial complex with vertices  $x_1, \dots, x_n$  and let  $I_\Sigma$  be the ideal of the polynomial ring  $R = K[x_1, \dots, x_n]$  generated by the square-free monomials that are products of a set of variables that is *not* a face of  $\Sigma$ . Prove the fact stated in class that the minimal primes of  $R/I_\Sigma$  are in bijective correspondence with the facets (maximal faces) of  $\Sigma$ , where the prime corresponding to  $\sigma \in \Sigma$  is generated by the image of  $\{x_1, \dots, x_n\} - \sigma$ .
3. (a)  $R, S$  are finitely generated  $\mathbb{N}$ -graded  $K$ -algebras with  $R_0 = S_0 = K$ , and  $T = R \otimes_K S$  is  $\mathbb{N}$ -graded by  $[R \otimes_K S]_d = \bigoplus_{i+j=d} R_i \otimes_K S_j$ . Prove:  $\mathfrak{P}_T(z) = \mathfrak{P}_R(z)\mathfrak{P}_S(z)$ .  
 (b) Let  $R = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$ , but graded so that  $\deg(x_i) = m_i$  for  $1 \leq i \leq n$ . (Note: when  $R$  is not necessarily generated by one-forms, the Hilbert-Poincaré series and the Hilbert function are defined, but the Hilbert function *need not* be eventually polynomial.) Prove that  $\mathfrak{P}(R) = \prod_{i=1}^n 1/(1 - z^{m_i})$ .
4. Recall that an  $R$ -module or algebra  $S$  is *flat* over  $R$  if  $S \otimes_R \_$  preserves exactness (preserving injectivity of maps of modules suffices), and *faithfully flat* over  $R$  if it is  $R$ -flat and  $M \neq 0 \Rightarrow S \otimes_R M \neq 0$  (this implies a sequence of  $R$ -modules is exact iff its tensor product with  $S$  is exact). Nonzero free modules are faithfully flat, and if  $K \subseteq L$  are fields,  $L$  is faithfully flat over  $K$ , since it is free. Prove the following statements:
  - (a) If  $x \in R$  is a nonzerodivisor on  $M$  and  $S$  is  $R$ -flat then  $x$  is a nonzerodivisor on  $S \otimes_R M$ .
  - (b) If  $x_1, \dots, x_n \in R$ ,  $M$  is an  $R$ -module, and  $S$  is faithfully flat over  $R$ , then  $x_1, \dots, x_n$  is a regular sequence on  $M$  iff it is a regular sequence on  $S \otimes_R M$ .
  - (c) If  $R$  is a finitely generated  $K$ -algebra and  $\dim(R) = n$ , so that by the Noether Normalization Theorem,  $R$  is a module-finite over a polynomial subring  $K[x_1, \dots, x_n] \subseteq R$ , then  $L[x_1, \dots, x_n] \subseteq L \otimes_K R$  is module-finite, and so  $\dim(L \otimes_K R) = n$ .
  - (d) If  $R$  is finitely generated and  $\mathbb{N}$ -graded over  $R_0 = K$  and  $F_1, \dots, F_n$  is a homogeneous system of parameters, then  $1 \otimes F_1, \dots, 1 \otimes F_n \in L \otimes_K R$  is a homogeneous system of parameters; moreover,  $L \otimes_K R$  is Cohen-Macaulay iff  $R$  is Cohen-Macaulay.
5. Let  $R = K[x_1, x_2, x_3, x_4]$  and let  $I = (x_1, x_2) \cap (x_3, x_4)$ . Find a homogeneous system of parameters for  $R/I$ , and verify that it is not a regular sequence in  $R/I$ . (It was asserted in class that Reisner's criterion implies that this ring is not Cohen-Macaulay. This problem gives an independent verification.)
6. Let  $I$  and  $J$  be monomial ideals in the polynomial rings  $R = K[x_1, \dots, x_n]$  and  $S = K[y_1, \dots, y_r]$  respectively such that the respective Hilbert-Poincaré series of  $R/I$  and  $S/J$  are  $F(z)$  and  $G(z)$ . Let  $T = K[x_1, \dots, x_n, y_1, \dots, y_r]$ , the polynomial ring in  $n + r$  variables over  $K$ . Prove that the Hilbert-Poincaré series of  $T/IJT$  is given by  $F(z)/(1 - z)^r + G(z)/(1 - z)^n - F(z)G(z)$ . (It may be helpful that  $IJT = IT \cap JT$ : if you use this, prove that it is true.)