Math 615, Fall 2007 **Problem Set #3**

Due: Friday, March 9

1. Let I be an ideal of a Noetherian ring R and let M be a finitely generated R-module such that $IM \neq M$. Prove that $\mathrm{depth}_I M = \inf\{\mathrm{depth}_{I_P} M_P : P \in \mathrm{Supp}\,(M/IM)\}$. Show that $\mathrm{depth}_I M$ cannot exceed the number of generators of I, and cannot exceed the Krull dimension of M. Also show that if $\mathrm{Rad}\,(I) = \mathrm{Rad}\,(J)$, then $\mathrm{depth}_I M = \mathrm{depth}_J M$.

2. (a) Let R be a Noetherian ring and let I be an ideal of R. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules for each of which depth on I is finite. Prove that if $\mathrm{depth}_I M > \mathrm{depth}_I M''$, then $\mathrm{depth}_I M' = \mathrm{depth}_I M'' + 1$.

(b) Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring with homogeneous maximal ideal $m = (x_1, \ldots, x_n)R$, and let M be a finitely generated nonzero $\mathbb{Z}$-graded module. Recall the class result that M is free if and only if $\mathrm{depth}_I M = n$. Show that if $\mathrm{depth}_m M = n - k < d$, then any graded kth module of syzygies of M is free, while if $0 \leq h < k$, no hth module of syzygies of M is free. (This is the graded form of the Hilbert Syzygy Theorem.)

3. (a) Let $\mathcal{B}$ be a well-ordered set (i.e., a totally ordered set with DCC). Let $\mathcal{F}$ be the set of finite subsets of $\mathcal{B}$, and put a total ordering on $\mathcal{F}$ as follows: $A > B$ if A contains B strictly or if $A = \{a_1, \ldots, a_k\}$ with $a_1 > \cdots > a_k$, $B = \{b_1, \ldots, b_h\}$ with $b_1 > \cdots > b_h$, and there exists $i \leq \min\{h, k\}$ such that $a_j = b_j$ for $j < i$ while $a_i > b_i$. (E.g., if $a > b > c$ then $\{a, b\} > \{a\}$, $\{a, b\} > \{a, c\}$, and $\{a\} > \{b, c\}$.) Prove that this well-orders $\mathcal{F}$.

(b) Let F be a finitely generated free $K[x_1, \ldots, x_n]$-module with ordered basis. Fix a monomial order on F. One can modify the non-deterministic division algorithm as follows: let $f \in F$ and $g_1, \ldots, g_r \in F$. At each step, if some $\mathrm{in}(g_i)$ divides some term of the "current" remainder f', choose *any* such term ν of f' (it need not be biggest) and subtract a scalar times a monomial multiple of g_i, say $c\mu g_i$, so as to cancel ν (add $c\mu$ to the coefficient of g_i, and subtract $c\mu g_i$ from f'). Prove that this process must terminate after finitely many steps.

4. A group G acts by ring automorphisms on a domain R. The action of G extends uniquely to an action on the fraction field $\mathcal{F}$ of R (you need not prove this.)

(a) Prove that if G is finite, then $\mathrm{frac}\,(R^G) = \mathcal{F}^G$.

(b) Let $G = K - \{0\}$ under $\cdot$ act on the polynomial ring $R = K[x, y]$ over the field K by letting $a : x \mapsto ax$ and $a : y \mapsto ay$. Is it true that $\mathrm{frac}\,(R^G) = \mathcal{F}^G$ in this case?

(c) Suppose that R is integrally closed in $\mathcal{F}$. Prove that R^G is integrally closed in its fraction field.

5. Let K be a field and let $X = \big(x_{ij}\big)$ be an $m \times n$ matrix of indeterminates over K, where $2 \leq m \leq n$. Let t be an integer such that $2 \leq t \leq m$. Let Y be an $(m-1) \times (n-1)$ of new indeterminates. Show that $\big(K[X]/I_t(X)\big)_{x_{mn}}$ is isomorphic with a localization of a polynomial ring over $K[Y]/I_{t-1}(Y)$. Hence, if x_{mn} is a nonzerodivisor on $K[X]/I_t(X)$ and $K[Y]/I_{t-1}(Y)$ is an integral domain, then $K[X]/I_t(X)$ is an integral domain.

6. Let notation be as in Problem 5. above with $t = 2$. Find a Gröbner basis for $I_2(X)$ in revlex, with the variables ordered so that $x_{hi} > x_{jk}$ if $h < j$ or $h = j$ and $i < k$. Explain why x_{mn} is not a zerodivisor on $K[X]/I_2(X)$. Prove that $I_2(X)$ is a prime ideal.