Math 615, Fall 2007 **Problem Set #4**
Due: Friday, March 23

1. Let K be a field, and let $X = \bigl(x_{i,j}\bigr)$ be a 2×3 matrix of indeterminates over K. Order the variables as in Problem 6. of the preceding assignment. Let Δ_i be the 2×2 minor obtained by deleting the j th column of X and taking the determinant. Use hlex and Schreyer's method to find the module N of relations on Δ_1, Δ_2, Δ_3. Prove that N has two minimal generators. Confirm in this way that the free resolution for $K[X]/I_2(X)$ in the last displayed line on p. 6 of the Lecture Notes of February 23 is correct.

2. Let R be an $\mathbb{N}$-graded ring that is finitely generated over a field $K = [R]_0$ and let S be an $\mathbb{N}$-graded, degree-preserving module-finite extension finitely generated over a field $L = [S]_0$. (L may be finite algebraic over K.) Prove: if $R \hookrightarrow S$ splits as a map of R-modules and S is Cohen-Macaulay, then R is Cohen-Macaulay. Prove that if R is Cohen-Macaulay then for every integer $k > 0$ the *Veronese subring* $R^{(k)} = \bigoplus_{d=0}^{\infty}[R]_{kd}$ is Cohen-Macaulay.

3. Let X and Y be 2×2 matrices of inderminates over a field K. Let $I_1(XY - YX)$ be the ideal generated by the entries of $XY - YX$. Prove that $K[X, Y]/I_1(XY - YX)$ is a Cohen-Macaulay domain. [Suggestion: this ring can be shown to be a polynomial ring over a domain that we already know to be Cohen-Macaulay.]

N.B. The same question may be asked for every size n. The corresponding ring is known to be a Cohen-Macaulay domain if $n = 3$ (M. Thompson). This is conjectured to be the case in general, but this is an open question.

4. Let G be a linear algebraic group over an algebraically closed field K acting on a K-algebra R so that R is a G-module. Let $H \subseteq G$ be a subgroup that is dense in G in the Zariski topology. Show that $R^H = R^G$.

Let H be the group of unitary matrices $\gamma \in \mathrm{GL}(n, \mathbb{C})$ (i.e., $\gamma \in H$ if its inverse is the transpose of its complex conjugate). Find, with proof, the Zariski closure of H in $\mathrm{GL}(n, \mathbb{C})$.

5. A ring R of prime characteristic $p > 0$ is called *F-split* if the Frobenius endomorphism $F : R \to R$ is injective and $F(R) = R^p$ is a direct summand of R as $F(R)$-modules.

(a) A domain R is called *seminormal* if whenever f is in the fraction field of R and f^2, $f^3 \in R$, then $f \in R$. Prove that an F-split domain is seminormal.

(b) Let K be a field of characteristic $p > 0$ which we assume, for simplicity, is perfect. Prove that the polynomial ring $R = K[x_1, \ldots, x_n]$ is F-split, and that R/I is F-split if I is generated by square-free monomials.

(c) Let R be a polynomial ring as in part (b), let G be a subgroup of the group of permutations of $x_1, \ldots, x_n$, and let G act on R by K-algebra automorphisms that extend its action on the set of variables. Must R^G be F-split? Prove your answer.

6. Consider the situation of problem 1. again, and suppose that K has characteristic $p > 0$. Show directly that for some $t > 0$ the ideal $(\Delta_1^t, \Delta_2^t, \Delta_3^t) \subseteq K[\Delta_1, \Delta_2, \Delta_3] = R$ is not contracted under the map $R \subseteq K[X] = S$, which implies that R is not a direct summand of S over R. (This can be deduced from the local cohomology argument given in the Lecture Notes from February 23, but you are being asked to find a different, more elementary argument.) Note that these ideals *are* contracted if K has characteristic 0.