Math 615, Fall 2007 **Problem Set #5**
Due: Monday, April 16

1. Let R be a Noetherian domain of prime characteristic $p > 0$, I, $J \subseteq R$, and let $f \in R$.

(a) Show that if I is tightly closed, then $I :_R J$ is tightly closed.

(b) Show that if $f \neq 0$ and $g \in (f)^*$, then g/f is integral over R. (You may assume the fact that the integral closure of R is an intersection of Noetherian discrete valuation rings.)

(c) Show that if R is integrally closed, then $(fI)^* = f(I^*)$.

2. Let (R, m, K) be a Cohen-Macaulay ring of dimension n. Let $\underline{x} = x_1, \ldots, x_n$ and $\underline{y} = y_1, \ldots, y_n$ be two systems of parameters. Let $V_{\underline{x}}$ be the annihilator of m in $R/(\underline{x})R$, with similar notation for $\underline{y}$ and other systems of parameters.

(a) If $n = 1$, prove that multiplication by y_1 on the numerators induces an injective map $R/x_1R \to R/x_1y_1R$ that carries $V_{x_1} \cong V_{x_1y_1}$.

(b) Show that $V_{\underline{x}} \cong V_{\underline{y}}$ in general. (Any two systems of parameters are joined by a finite chain in which consecutive systems differ in only one element.)

(c) If $y_i = x_i^{t_i}$ with $t_i \geq 1$ for $1 \leq i \leq n$, show that the map $R/(\underline{x})R \to R/(\underline{y})R$ induced by multiplication by $x_1^{t_1-1} \cdots x_n^{t_n-1}$ on numerators induces an injection that carries $V_{\underline{x}} \cong V_{\underline{y}}$.

[$\dim_K(V_{\underline{x}})$ is independent of the choice of $\underline{x}$ and is called the *type* of the Cohen-Macaulay local ring R. A Cohen-Macaulay local ring of type one is called *Gorenstein*.]

3. Let (R, m, K) be a Cohen-Macaulay local domain of prime characteristic $p > 0$. Let $x_1, \ldots, x_n$ be a system of parameters. Suppose that $(x_1, \ldots, x_n)R$ is tightly closed. Prove that for every system of parameters $y_1, \ldots, y_n$, the ideal $(y_1, \ldots, y_n)R$ is tightly closed.

4. Fix $n \geq 2$ and integers $a_1, \ldots, a_n > 0$. Let K be a field of characteristic $p > 0$ (p will vary) and let $f = x_1^{a_1} + \cdots + x_n^{a_n}$. Let $R = K[x_1, \ldots, x_n]/(f)$, and let $I = (x_1, \ldots, x_{n-1})R$. Let $\alpha = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$. Show that if $\alpha \leq 1$, then $x_n^{a_n-1}$ is in the tight closure I^* of I. Show that if $\alpha > 1$ then for all sufficiently large primes p, the element $x_n^{a_n-1} \notin I^*$.

5. Let R be a complete local ring of mixed characteristic p.

(a) Show that if p is not part of a system of parameters, i.e., if $\dim(R/pR) = \dim(R)$, then R is not a module-finite extension of a regular local ring.

(b) Show that if $\mathrm{Ass}(R)$ contains only minimal primes P of R and for every such P, $\dim(R/P) = \dim(R)$, then R is a module-finite extension of a ring of the form $T/(f)$, where $T = V[[X_1, \ldots, X_n]]$ is a formal power series ring over a complete DVR (V, pV) and $f \neq 0$. You may assume that T is a unique factorization domain.

6. Let K be a field in which -1 is not a square, let t be a variable over K, let $P = (t^2 + 1)K[t]$, which is prime in $K[t]$, and let $(S, m, L) = K[t]_P$. Note that $L \cong K[\sqrt{-1}]$.

(a) First, suppose that $K = \mathbb{R}$, so that $L = \mathbb{C}$. It was shown in class that S has no coefficient field. Find an explicit power series in powers of $t^2 + 1$ with terms in S that converges to a square root of -1.

(b) Is it true for all fields K such that -1 is not a square in K that S has no coefficient field? (Note that a coefficient field need not contain the "obvious" copy of K in S.)