Math 615, Fall 2010 **Problem Set #1**

Due: Friday, February 5

1. Let K be a field and let $M = \{x_1, x_2, \ldots, x_n, \ldots\} \cup \{1\}$ be a countably infinite totally ordered set such that $x_1 < x_2 < \cdots < x_n < \cdots < 1$. (E.g., one may take $M \subseteq \mathbb{Q}$ with $x_i = i/(i{+}1)$, $i \geq 1$.) M is a commutative semigroup with identity 1 if $mm' = \min\{m, m'\}$. Consider the semigroup ring R of M over K: this is a K-vector space with basis M such that multiplication of elements of M is given by the semigroup operation. Thus, if $i \leq j$ then $x_i x_j = x_i$ in R. In fact you may assume that $R = K[X_i : i \geq 1]/(X_i - X_i X_j : i \leq j)$, where the X_i are indeterminates. Prove that the ideal $m = (x_i : i \geq 1)R \subseteq R$ is both maximal and minimal among the primes of R, but that $\{m\}$ is not a clopen set in $\mathrm{Spec}\,(R)$.

EXTRA CREDIT 1. Give an explicit description of $\mathrm{Spec}\,(R)$ as a topological space.

2. Let $m, n > 0$ be relatively prime integers. Let $R \hookrightarrow R[u] = S$ be a ring extension such that u^m and u^n are in R. Show that $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ is bijective.

3. (a) Let $h : (R, m) \to (S, n)$ be a homomorphism of local rings that is local, i.e., $h(m) \subseteq n$. Prove that $\dim(S/mS) \geq \dim(S) - \dim(R)$.

In the remaining problems, let K be an algebraically closed field and let $f : X \to Y$ be a dominant morphism of closed affine algebraic varieties over K. Let $h : R = K[Y] \subseteq K[X] = S$ be the induced inclusion of integral domains.

3. (b) Suppose that $y = f(x)$ is in the mage of f. Show that the dmension of the fiber $Z = f^{-1}(y)$ over y is $\geq \delta = \dim(X) - \dim(Y)$.

4. Prove or disprove: f is onto with finite set-theoretic fibers iff S is a module-finite extension of R.

5. Assume that f is bijective and that $K(X)$ is separable over $K(Y)$. Prove or disprove that S is contained in the integral closure of R in its fraction field.

6. Let $K[x, y]$ be the coordinate ring of $\mathbb{A}^2_K$, and let Y be the variety $\mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2_K$, whose coordinate ring $R = K[Y]$ is $\cong K[x, y]/(y^2 - x^2 - x^3)$.

(a) Prove that the K-algebra map $R \to K[t]$ such that $x \mapsto t^2 - 1$ and $y \mapsto (t^2 - 1)t = t^3 - t$ is an isomorphism. Hence, $R \cong K[t^2 - 1, t^3 - t] \subseteq K[t]$.

(b) What is the integral closure T of R in $K(Y)$? How is it related to $K[t]$?

(c) What is the K-vector space dimension of T/R?

(d) What is the conductor $\mathfrak{A} \subseteq R$ of $R \hookrightarrow T$?

EXTRA CREDIT 2. Let $n \geq 2$ be an integer. If A is an $n \times n + 1$ matrix, let A_j be the $n \times n$ submatrix obtained by deleting the j th column of A, and let $D_j(A) = \det(A_j)$. Let X be the variety whose points correspond to the $n \times (n + 1)$ matrices over K (a choice basis gives identifies this K-vector space with $\mathbb{A}_K^{n(n+1)}$). Let $f : X \to \mathbb{A}_K^{n+1} = Y$ by $A \mapsto (D_1(A), \ldots, D_{n+1}(A))$. Show that the dimension of the fiber $f^{-1}(y) \subseteq X$ over any $y \in \mathbb{A}^{n+1}$ except 0 is $n^2 - 1$, and determine the dimension (which is $> n^2 - 1$) of $f^{-1}(0)$.