Math 615, Fall 2010 **Problem Set #2**

Due: Friday, February 19

1. Let $R \to S$ be formally smooth (respectively, formally unramifed, respectively formally étale). Let T be an R-algebra and J an ideal of T. Let $n \geq 2$ be a positive integer. Suppose that $J^n = (0)$. Show that every R-algebra homomorphism $S \to T/J$ lifts (respectively, has at most one lifting, respectively, has a unique lifting) to a homomorphism $S \to T$.

2. Let K be a field. Let $X = X_1, X_2, \ldots, X_n$ be indeterminates over K.

(a) Let K have characteristic $p > 0$. Let R be a K-algebra, and let $F_1, \ldots, F_n$ be polynomials over R whose linear terms $L_1, \ldots, L_n$ have coefficients in K. Suppose that in every nonlinear term of every F_j, every exponent that occurs on one of the X_i is a multiple of p. Show that if $L_1, \ldots, L_n$ are linearly independent over K, then $R[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$ is étale over R.

(b) Let K have characteristic 0. Let c be an element of K. Give a simple condition on c and h characterizing when $K[X]/(X^h - X + c)$ is étale over K.

3. Let S_1 and S_2 be R-algebras.

(a) Show that if S_1 and S_2 are both formally smooth, formally unramified, or formally étale over R, then so is $S = S_1 \otimes_R S_2$.

(b) Show that $\Omega_{S/R} \cong S_2 \otimes_R \Omega_{S_1/R} \oplus S_1 \otimes_R \Omega_{S_2/R}$.

4. Let I be a finitely generated ideal of a ring R such that $I = I^2$. Prove that I is generated by an idempotent. Conclude that a finitely generated R-algebra S is formally unramified iff $\mathrm{Ker}\,(S \otimes_R S \to S)$ (where $s \otimes s' \mapsto ss'$) is generated by an idempotent.

5. Prove carefully, directly from the definition of formally étale, the assertion made in class that $R \to S$ is formally étale if and only if $R \to S_Q$ is formally étale for all $Q \in \mathrm{Spec}\,(S)$. Also show that if $R \to (S, Q)$ is a homomorphism to a quasilocal ring (S, Q) and P is the contraction of Q to R, then $R \to S$ is formally smooth or formally unramifed iff $R_P \to S_Q$ has the corresponding property.

6. Let k be a perfect field of characteristic $p > 2$, let $K = k(u, v)$, where u and v are indeterminates, and $L = K[y]/(y^{2p} + uy^p - v)$. Show that L is a finite algebraic extension of K that is not separable, but that L does not meet $K^{1/p} - K$.

EXTRA CREDIT 3. Let $S = R[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$ be a presentation of S over R, and let $\mathcal{J}$ be the image of $\big(\partial F_j/\partial X_i\big)$ in S, which defines $\theta : S^m \to S^n$. Show that S is unramified over R (respectively, smooth over R) if for every S-module M, the map $M^m \to M^n$ induced by applying $M \otimes_S \_$ is onto (respectively, one-to-one).

EXTRA CREDIT 4. Characterize when an $n \times m$ matrix $\big(s_{ij}\big)$ over S has the property that the map $M^m \to M^n$ that it induces is one-to-one (respectively, onto) for every S-module M without referring to the modules M.

EXTRA CREDIT JC. Let R be the polynomial ring in two variables over $\mathbb{C}$. Is there an étale $\mathbb{C}$-algebra map $R \to R$ that is not an isomorphism? What if R is the polynomial ring in n variables over $\mathbb{C}$, $n \geq 2$?