Math 615, Fall 2010 **Problem Set #3**
Due: Friday, March 12

1. Prove that a direct limit of formally unramified (respectively, formally étale) R-algebras is formally unramified (respectively, formally étale).

2. Show that a direct limit of Henselian quasilocal rings in which the homomorphisms are local is Henselian.

3. Let K be a field of characteristic 0, and construct an increasing sequence of K-algebras R_n as follows. Let $X = X_1$ be an indeterminate, and let $R_1 = K[X_1]$. Recursively, suppose that $R_n = K[X_n]$ is a polynomial ring in one variable over K, $n \geq 1$, and let $R_{n+1} = R_n[X_{n+1}]/(X_{n+1}^2 - X_n)$. (One may think of X_n as $X^{1/2^{n-1}}$.) Let $R = \bigcup_{n=1}^{\infty} R_n$, a directed union of polynomial rings in one variable. Determine whether R is formally smooth over K. Prove your answer.

4. Let $K_1, \ldots, K_n, \ldots$ be fields, and let $R = \prod_{n=1}^{\infty} K_n$ be their product.

(a) Show that every element of R is the product of a unit and an indempotent, and show that R is a zero-dimensional ring, i.e., that every prime ideal is maximal.

(b) Suppose all of the K_n are K-algebras, where K is a field, and assume as well either that (1) infinitely many K_n are infinite fields or (2) there is an infinite set of K_n of finite cardinality such that the cardinals of these K_n are not bounded. ((1) is immediate if K is infinite.) Show that R contains an element that is transcendental over K.

(c) Suppose also that K has characteristic 0 (e.g., this holds if $K = K_n = \mathbb{Q}$ for all n). Show that R is *not* formally unramified over K.

5. Let (R, m, K) be a Henselian ring. Let $Z_1, \ldots, Z_n$ be indeterminates over R: the subscripts should be read modulo n. Let $u_1, \ldots, u_n \in m$ and let $r_1, \ldots, r_n \in R$. Let $h_1, \ldots, h_n$ be integers all of which are ≥ 2.

Show that the n simultaneous equations
$$\begin{aligned} u_1 Z_1^{h_1} + Z_2 &= r_1 \\ &\cdots \\ u_i Z_i^{h_i} + Z_{i+1} &= r_i \\ &\cdots \\ u_n Z_n^{h_n} + Z_1 &= r_n \end{aligned}$$
have a solution in R. (One approach is to make use of a suitable pointed étale extension.)

6. Let $A = K[[x, y]]$ be the formal power series ring in two variables over a field K. Let $P = xA$, which is a prime ideal. Prove that the local ring R_P is *not* Henselian by showing that there is a monic polynomial $F = F(Z)$ in one indeterminate over R such that when F is considered modulo PR_P it has simple roots, but such that F has no root in R.

EXTRA CREDIT 5. Show that there exists an algebra S over a Noetherian ring R of prime characteristic $p > 0$ such that S is module-finite and free as an R-module and $\Omega_{S/R}$ is free as an S-module, but S is not smooth over R. (This shows that in characteristic p, one needs a further condition, e.g., on the rank of $\Omega_{S/R}$, in order to conclude that S is smooth over R.)