Math 615, Fall 2010 **Problem Set #4**
Due: Wednesday, April 7

1. (a) Show that if S is smooth over a quasilocal ring $(R,\, P)$, then R is a direct limit of local Noetherian subrings $(R_\lambda,\, P_\lambda)$ and local maps, and that one can choose one of these, say R_0, and a smooth extension S_0 of R_0 such that $S = S_0 \otimes_{R_0} R$. Hence, S is the direct limit of the rings $S_0 \otimes_{R_0} R_\lambda$ for $\lambda \geq \lambda_0$.

(b) Let $(R,\, P) \to (S,\, Q)$ be a flat local homomorphsim of quasilocal rings such that $(R,\, P)$ is reduced. Suppose that R has only finitely many minimal primes (which holds, for example if R is Noetherian), and that the fiber over every minimal prime is reduced. Show that S is reduced.

(c) Show that if R is reduced and S is essentially smooth over R, then S is reduced.

(d) Show that if $(R,\, P)$ is quasilocal and reduced, then its Henselization is reduced.

2. Let R be the localization of a domain of Krull dimension one finitely generated over the complex numbers $\mathbb{C}$. Show by example that the Henselization of R need not be a domain.

3. Let (R, P, K) be a quasilocal ring and let M be an $s \times s$ matrix over R that is congruent to the identity matrix mod P. Prove that if n is a positive integer not divisible by the characteristic of K, then M has an nth root over a pointed étale extension of R.

4. Let (R, P, K) be a quasilocal ring and let N be the ideal of all nilpotent elements of R. Suppose that R/N is Henselian. Prove or disprove that R must be Henselian.

5. Let $R \to S$ be local homomorphism of quasilocal rings such that S is a module-finite extension of R. Prove that $S^{\mathrm{h}} \cong R^{\mathrm{h}} \otimes_R S$, or give a counterexample.

6. Let $p > 0$ be a prime integer. If A is a ring of characteristic p, let $F_A : A \to A$ denote the Frobenius endomorphism, i.e., $F_A(a) = a^p$ for all $a \in A$. Let $R \to S$ be a homomorphism of rings of characteristic p. Show that:

(a) if $F_S : S \to S$ is surjectve, then S is formally unramified over R, and

(b) if F_R is surjective and F_S is an automorphism, then S is formally étale over R.

EXTRA CREDIT 6. Must a formally étale algebra S over a field K be reduced? Prove this, or give a counterexample.

EXTRA CREDIT 7. Let R be Noetherian and formally smooth over a perfect field K. Prove that R is regular. [Suggestion: reduce to the local case, (R, P). Note that a perfect subfield of a complete local ring is always contained in a coefficient field. (This is clear in characteristic 0. In characteristic $p > 0$, any perfect field $\kappa \subseteq R$ is contained in *every* coefficient field. Cf. the Theorem on p. 12 of the supplement on *The structure theory of complete local rings*: $\kappa = \kappa^{p^n} \subseteq R^{p^n} \subseteq R_n$.) Thus, one has a coefficient field $L \subseteq R/P^2$ (which is complete) with $K \subseteq L$. Let $x_1,\, \ldots\,, x_d$ be a minimal set of generators of P. Then we have a surjection $R \twoheadrightarrow R/P^2 \cong L[X_1,\, \ldots\,, X_d]/m^2$, where $\underline{X} = X_1,\, \ldots\,, X_d$ are indeterminates and $m = (\underline{X})$. Use that R is formally smooth over K to show that this lifts to a map $R \twoheadrightarrow L[\underline{X}]/m^{n+1}$ for all n, whence $\ell(R/P^{n+1}) \geq \ell(L[\underline{X}]/m^{n+1})$. Conclude that $\dim(R) \geq d$.]