Math 615, Fall 2010 **Problem Set #5**
Due: Wednesday, April 21

1. Prove that if R and S are separated K-algebras, then $R \otimes_K S$ is separated.

2. Prove that if $(R, m.K)$ is a local ring that is an approximation ring then
(a) if R is a domain, so is $\widehat{R}$ and (b) if R is reduced, so is $\widehat{R}$.

3. Let T be a formally smooth R-algebra, $I \subseteq T$ an ideal, and let $S = T/I$. Let $\bar{\ }$ indicate images mod I^2, and $[\ ]$ mod I or mod $I\Omega_{T/R}$. The restriction of $d : T \to \Omega_{T/R}$ to I gives a map $I \to \Omega_{T/R}$ and, hence, $I \to \Omega_{T/R}/I\Omega_{T/R} \cong S \otimes_T \Omega_{T/R}$.
(a) Show that the map described kills I^2, and so induces a map $\eta : I/I^2 \to \Omega_{T/R}/I\Omega_{T/R}$ such that if $f \in I$, $\overline{f} \mapsto [df]$. Prove also that η is S-linear.
We know that for any surjection $T \to T/I = S$ of R-algebras, Ω_S is the quotient of $S \otimes_T \Omega_{T/R}$ by the R-span of the image of $\{df : f \in I\}$. Hence, there is an exact sequence $I/I^2 \xrightarrow{\eta} \Omega_{T/R}/I\Omega_{T/R} \to \Omega_{S/R} \to 0$ of S-modules.
(b) Show that giving a splitting, $\phi : T/I \to T/I^2$, as R-algebras, of the map $T/I^2 \twoheadrightarrow T/I$, is equivalent to giving, for all $t \in T$, an element $\delta(t) \in I/I^2$ such that $\phi([t]) = \bar{t} - \delta(t)$, subject to the conditions that $\delta : T \to I/I^2$ be an R-derivation and that $\delta(f) = \overline{f}$ for all $f \in I$. Thus, δ corresponds to a T-linear map $\theta : \Omega_{T/R} \to I/I^2$ such that $\delta = \theta \circ d$, and for all $f \in I$, $\theta\big(\eta(\overline{f})\big) = \overline{f}$. Show also that if $f \in I$ and $t \in T$, $\theta(f dt) = 0$, so that θ induces an S-linear map map $\Omega_{T/R}/I\Omega_{T/R} \to I/I^2$.
(c) Conclude that S is formally smooth over R iff $\eta : I/I^2 \to \Omega_{T/R}/I\Omega_{T/R}$ splits as a map of S-modules (hence, also, iff η is injective and $\Omega_{T/R}/\Omega_{T/R} \twoheadrightarrow \Omega_{S/R}$ splits over S).

4. Let K be a field, and let $R = K[X_1, X_2, X_3, \ldots, X_n, \ldots]$ be the polynomial ring in countably many variables over K, let $m = (X_n : n \geq 1)R$, let S by the m-adic completion of R, and let $f_n = \sum_{j=n}^{\infty} x_j^j$. Is $f_n \in m^n S$? Prove your answer.

5. Let (A, m, K) be a complete local ring and let $R = A[[X_1, \ldots, X_n]]$ be a formal power series ring over A. Let $f_1, \ldots, f_n$ be elements of the maximal ideal of R whose images in $R/mR \cong K[[X_1, \ldots, X_n]]$ form a system of parameters. Prove that $R/(f_1, \ldots, f_n)R$ is module-finite over A.

6. Let D be a principal ideal domain such that $p_1 D, \ldots, p_n D \ldots$ is an enumeration of the distinct maximal ideals of D (e.g., one may take $D = \mathbb{Z}$ or $\mathbb{Q}[x]$). Let G be the free D-module with free basis $e_1, \ldots, e_n, \ldots$, and let $f : G \to G$ be such that $f(e_n) = e_n - p_n e_{n+1}$ for all n. Let $C = \operatorname{Coker}(f)$, so that $(*)\ 0 \to G \xrightarrow{f} G \to C \to 0$ is exact. Let L be the fraction field of D. Let W be the set of square-fee elements of D. Show that $C \cong \{a/b : a \in D, b \in W\} \subseteq L$. Show that $(*)$ is locally split but not split.

EXTRA CREDIT 8. Let D be as in #6. and let $T = K[X_n : n \geq 1]$ be a polynomial ring in a countably infinite set of variables over D. Let $I = (X_n - p_n X_{n+1} : n \geq 1)T$, and $S = T/I$. Prove that S_Q is formally smooth over D for all $Q \in \operatorname{Spec}(S)$: in fact S_P is smooth over D_P for all $P \in \operatorname{Spec}(D)$. Prove that S is not formally smooth over D.