The local nature of an element of a ring or module

Let R be any commuative ring, and let M be any R-module. Very possibly, $M = R$. Consider a Zariski open cover of $X = \mathrm{Spec}\,(R)$ by sets of the form $D(f_i) = \{P \in \mathrm{Spec}\,(R) : f_i \notin P\}$, Here, i runs through an index set I that may be infinite. If $u \in M$, then u has an image $u/1 = u_i$ in each of the modules M_{f_i}. Moreover, the image of u_i in $M_{f_if_j} \cong (M_{f_i})_{f_j}$ is the same as the image of u_j in $(M_{f_j})_{f_i}$: both are the same as the image of u in $M_{f_if_j}$. The theorem below provides a converse. This is an extraordinarily useful fact: roughly speaking, one may construct an element of a ring or module by constructing it locally with respect to an open cover, provided that the choices fit together on overlaps. This is one of the ideas that underlies the theory of schemes. We give a completely elementary proof of this result here, which does not need any knowledge of sheaves.

Theorem. *Let R be a ring, M and R-module, and let $\{D(f_i) : i \in I\}$ be a Zariski open cover of $X = \mathrm{Spec}\,(R)$. Suppose that for every $i \in I$, one is given $u_i \in M_{f_i}$ in such a way that for all i, $j \in I$, the image of u_i in $(M_{f_i})_{f_j}$ is the same as the image of u_j in $(M_{f_j})_{f_i}$ under the canonical identification of these two modules. Then there is a unique element $u \in M$ such that the image of u in M_{f_i} is u_i for every i.*

Proof. The $D(f_i)$ cover X if and only if the f_i generate the unit ideal, in which case finitely many cover. If there is a unique u for every finite cover, all of these elements u must be the same: given two finite covers, we may choose u to be the element that solves the problem for their union, and this element works for each of the original finite covers. The same element then solves the problem for the original (possibly infinite) cover.

Hence, we may assume without loss of generality that the cover is finite, with n open sets, $D(f_1), \ldots, D(f_n)$. Note that for positive integers k_i, $f_1, \ldots, f_n$ generate the unit ideal iff $f_1^{k_1}, \ldots, f_n^{k_n}$ generate the unite ideal. In fact, $D(f) = D(f^k)$ for $k > 0$. To prove uniqueness, suppose that u and u' have the same image in M_{f_i} for all i. Then $u - u'$ maps to 0 in each, which means that $f_i^{k_i}(u - u') = 0$ for some k_i. Thus, $\mathrm{Ann}_R(u - u') \supseteq (f_1^{k_1}, \ldots, f_n^{k_n}) = R$, and $u - u' = 0$.

To prove existence we use induction on n. If $n = 1$, f_1 is a unit, $M_{f_1} = M$, and the result is obvious.

We next consider the case where $n = 2$, which is the heart of the argument. We have elements $u_1 = w_1/f_1^h \in M_{f_1}$ and $u_2 = w_2/f_2^k \in M_{f_2}$, where w_1, $w_2 \in M$ and their images in $M_{f_1f_2}$ agree. This means that for some sufficiently large N, $(f_1f_2)^N(f_2^k w_1 - f_1^h w_2) = 0$, i.e., $f_2^{N+k}(f_1^N w_1) - f_1^{N+h}(f_2^N w_2) = 0$. Then $u_1 = f_1^N w_1/f_1^{N+h}$, and $u_2 = f_2^{N+k} w_2/f_2^{N+h}$. We may replace f_1, f_2 by f_1^{N+h}, f_2^{N+k}, respectively, and w_1, w_2 by $f_1^N w_1$, $f_2^N w_2$, respectively, to simply notation. Then, after this change, we may assume that $u_1 = w_1/f_1$, $u_2 = w_2/f_2$, and $f_2w_1 - f_1w_2 = 0$. The new f_1 and f_2 still generate the unit ideal, so that we have r_1, $r_2 \in R$ such that $r_1f_1 + r_2f_2 = 1$. Let $u = r_1w_1 + r_2w_2$. The image of u in M_{f_1} is $r_1w_1/1 + (r_2/f_1)(f_1w_2)/1 = (r_1f_1/f_1)w_1/1 + (r_2/f_1)(f_2w_1/1) = \big((r_1f_1 + r_2f_2)/f_1\big)w_1/1 = w_1/f_1 = u_1$. The image of u in M_{f_2} is u_2 by symmetry.

We now assume the result for a given $n \geq 2$, and prove existence for the case where we have $n+1$ open sets, $D(f_1), \ldots, D(f_{n+1})$ in the cover. Since the f_i generate the unit ideal, we can choose $r_1, \ldots, r_{n+1} \in R$ such that $\sum_{j=1}^{n+1} r_j f_j = 1$. Let $g = g_1 = \sum_{j=1}^{n} r_j f_j$ and let $g_2 = f_{n+1}$. In the ring R_g, the images $f_j/1$ of the elements $f_1, \ldots, f_n$ generate the unit ideal. We apply the induction hypothesis to the module M_g over this ring and the n open sets $D(f_1/1), \ldots, D(f_n/1)$. In $(M_g)_{f_j} \cong M_{gf_j} \cong (M_{f_j})_g$ we have the element $u_j/1$, the image of u_j when we localize at g. Moreover, for $1 \leq, i,\, j \leq n$, $u_i/1$ and $u_j/1$ have the same image in $M_{(f_i g)(f_j g)} = M_{f_i f_j g}$, since u_i and u_j have the same image in $M_{f_i f_j}$. Hence, there is a unique element $v_1 \in M_g$ whose images in the $M_{f_j g}$, $1 \leq j \leq n$, are the same as the images of the respective u_j. Let $v_2 = u_{n+1} \in M_{g_2}$. Since $g_1 + r_{n+1} g_2 = 1$, we have that g_1 and g_2 generate the unit ideal in R. We next show that v_1 and v_2 have the same image in $M_{g_1 g_2}$. Let w denote the difference of the images. Since $f_1/1, \ldots, f_n/1$ generate the unit ideal in R_{g_1}, and $M_{g_1 g_2}$ is a module over R_{g_1}, it suffices to show that every $f_j/1$ has a power that kills w. Thus, it suffices to show that v_1 and v_2 have the same image in every $M_{f_j g_2 g}$, $1 \leq j \leq n$. We know that the image of v_1 is the same as the image of u_j, while the image of v_2 is the image of u_{n+1}. But u_j and u_{n+1} have the same image even in $M_{f_j f_{n+1}} = M_{f_j g_2}$ by hypothesis., and this remains true when we localize further.

By the case where $n = 2$, there exists an element $u \in M$ whose image in M_{g_1} is v_1 and whose image in $M_{g_2} = M_{f_{n+1}}$ is $v_2 = u_{n+1}$. It remains to show that the image of u in M_{f_j} is u_j for $1 \leq j \leq n$. Since g_1 and g_2 generate the unit ideal in R, it suffices to prove this after localization at g_1 and at g_2. But the image of u in $M_{f_j g_1}$ is the same as the image of u_j by construction, and the image of u in $M_{f_j g_2}$ is the image of $u_{n+1}/1$ in $M_{f_j f_{n+1}}$, which is the same as the image of u_j by hypothesis. $\square$