Regular rings and finite projective resolutions

Recall that a local ring $(R,\, m,\, K)$ is *regular* if its embedding dimension, $\dim_K(m/m^2)$, which may also be described as the least number of generators of the maximal ideal m, is equal to its Krull dimension. This means that a minimal set of generators of m is also a system of parameters. Such a system of parameters is called *regular*. Another equivalent condition is the the associated graded ring of R with respect to m be a polynomial ring, in which case the number of variables is the same as $\dim(R)$. A regular local ring is a domain.

The following fact about regular local rings comes up frequently.

Proposition. *Let $(R,\, m,\, K)$ be a regular local ring. Let $J \subseteq m$ be a proper ideal of R. Then R/J is regular if and only if J is generated by part of a minimal set of generators for m, i.e., part of a regular system of parameters. (This is true if $J = 0$, since we may take the set to be empty.)*

Proof. If J is generated by $x_1,\, \ldots\,, x_k$, part of a minimal set of generators for m, then x_1 is not in an minimal prime, since R is a domain, and both the dimension and the embedding dimension of R/x_1R are one less than the corresponding number for R. It follows that R/x_1R is again regular, and the full result follows by a straightforward induction on k.

To prove the other direction, we also use induction on $\dim(R)$. The case where $\dim(R) = 0$ is obvious. Suppose $\dim(R) > 0$ and $0 \neq J \subseteq m^2$. Then R/J is not regular, for its dimension is strictly less than that of R, but its embedding dimension is the same. Thus, we may assume instead that there exists an element $x_1 \in J$ with $x_1 \notin m^2$, so that x_1 is part of a minimal set of generators for m. Then R/x_1R is again regular, and $(R/x_1R)/(J/x_1R) \cong R/J$ is regular. It follows that J/x_1R is generated by part of a minimal system of generators $\overline{x}_2,\, \ldots\,, \overline{x}_k$ for m/x_1R, where $\overline{x}_j$ is the image in R/x_1R of $x_j \in m$, $2 \leq j \leq k$. But then $x_1,\, \ldots\,, x_k$ is part of a minimal set of generators for m. $\square$

Projective resolutions

We want to characterize regular local rings in terms of the existence of finite free resolutions. Note that over a local ring, a finitely generated module is flat iff it is projective iff it s free.

Over any ring R, every module has a projective resolution. That is, given M, there is a (usually infinite) exact sequence $\cdots \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$, such that all of the G_i are projective. In fact, we may take them all to be free.

One can construct a free resolution as follows. First choose a set of generators $\{u_\lambda\}_{\lambda\in\Lambda}$ for M, and then map the free module $G_0 = \bigoplus_{\lambda\in\Lambda} Rb_\lambda$ on a correspondingly indexed set of generators $\{b_\lambda\}_{\lambda\in\Lambda}$ onto M: there is a unique R-linear map $G_0 \twoheadrightarrow M$ that sends $b_\lambda \mapsto u_\lambda$

for all $\lambda \in \Lambda$. Whenever we have such a surjection, the kernel M_1 of $P \twoheadrightarrow M$ is referred to as a *first module of syzygies* of M. If $0 \to M_1 \to P_0 \to M \to 0$ is exact with P_0 free, we may repeat the process and form an exact sequence $0 \to M_2 \to P_1 \to M_1 \to 0$. Then the sequence $0 \to M_2 \to P_1 \to P_0 \to M \to 0$ is also exact, where the map $P_1 \to P_0$ is the composition of the maps $P_1 \twoheadrightarrow M_1$ and $M_1 \hookrightarrow P_0$.

Recursively, we may form short exact sequences $0 \to M_n \to P_{n-1} \to M_{n-1} \to 0$ for all $n \geq 1$ (where $M_0 = M$), and then one has that every $n \geq 1$, the sequence

$$(*) \quad 0 \to M_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact. A module M_n that occurs in such an exacct sequence $(*)$ in which all the P_i are projective modules is called an n th module of syzygies of M. Equvialently, an n th module of syzygies may be defined recursively as a first module of syzygies of any $n-1$ st module of syzygies. Note that the (usually infinite) sequence

$$(**) \quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact as well, and so is a projective resolution of M.

A projective resolution is called *finite* if $P_n = 0$ for all $n \gg 0$. If M has a finite projective resolution, it is said to have *finite projective dimension.* The projective dimension of M is defined to be -1 if $M = 0$ and to be 0 if M is nonzero and projective. In general, if M has finite projective dimension and is not projective, the projective dimension n of M, which we denote $\mathrm{pd}_R M$ (or simply $\mathrm{pd}\, M$ if R is understood from context), is the smallest integer n for which one can find a finite projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$. If M does not have finite projective dimension, it is said to have infinite projective dimension, and one may write $\mathrm{pd}\, M = \infty$.

If R is Noetherian and M is finitely generated, one may also construct a module of syzygies by mapping a finitely generated free module onto M. The first module of syzygies will then be a submodule of this finitely generated free module, and, hence, finitely generated again. Therefore, M has a free resolution by finitely generated free modules.

In the sequel, we prove the following result:

Theorem. *Let (R, m, K) be a local ring. Then the following conditions are equivalent.*
(1) *R is regular.*
(2) *The residue field $K = R/m$ has a finite free resolution.*
(3) *Every finitely generated R-module has a finite free resolution.*

Before giving a proof, which will be based on elementary properties of Tor, we note an important consequence of this characterization of regularity.

Corollary. *If R is a regular local ring Q is a prime ideal of R, then R_Q is regular.*

Proof. Since R is regular, R/Q has a finite R-free resolution by R-modules. We may then localize at Q to obtain a finite R_Q-free resolution of the residue class field $R_Q/QR_Q \cong (R/Q)_Q$. $\square$

Thus, a Noetherian ring has the property that its localization at every prime ideal is regular if and only if it has the property that its localization at every maximal ideal is regular. A Noetherian ring with these equivalent properties is called *regular*.

Minimal free resolutions over local rings

Let (R, m, K) be local. We keep the notation of the preceding section. In constructing a free resolution for a finitely generated R-module M, we may begin by choosing a minimal set of generators for M. Then, at every stage, we may choose a minimal set of generators of M_{n-1} and use that minimal set to map a free R-module onto M_{n-1}. A resolution constructed in this way is called a *minimal* free resolution of M. Thus, a free resolution $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ iof M is *minimal* precisely if every P_n that occurs has a free basis that maps to a minimal set set of generators of the image M_n of P_n.

Our discussion shows that minimal free resolutions exist. We also note the following fact: a free resolution $\to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0$ (where the P_j are finitely generated free R-modules) is minimal if and only if for all $n \geq 1$, the image of P_j in P_{j_1} is contained in mP_{j-1}. The reason for this is that image of P_j consists of elements of P_{j-1} that give generators for the relations on the generators of M_{j-1}. These generators will be minimal generators if and only if they have no relation with a coefficient that is a unit, i.e., all of the generating relations are in mP_{j-1}. An equivalent way to phrase this is that entries of matrices for the maps $P_j \to P_{j-1}$ have all of their entries in m.

Recall that $\mathrm{Tor}_n(M, N)$ may be defined as the homology module at the n th spot of the complex $P_\bullet \otimes_R N$, where $P_\bullet$ is the complex $\cdots \to P_n \to \cdots \to P_0 \to 0$ obtained by replacing M by 0 in a projective resolutons $\cdots \to P_n \to \cdots \to P_0 \to M \to 0$. It is is independent of the specific projective resolution chosen up to canonical isomorphism. We assume familiarity with a few basic properties of Tor over a ring R as described in the supplement entitled *Exact sequences with a flat cokernel and a sketch of properties of Tor*. The specific facts that we need about Tor^R (we frequently omit the superscript) are these:

(1) If $\cdots \to P_n \to \cdots \to P_0 \to M$ is a projective resolution of M, then $\mathrm{Tor}_n(M, N)$ is the homology of the complex $\cdots \to P_n \otimes N \to \cdots \to P_0 \otimes N \to 0$ at the $P_n \otimes N$ spot. Hence, $\mathrm{Tor}_n(M, N) = 0$ if $n > \mathrm{pd}\, M$.
(2) $\mathrm{Tor}_n(M, N) = 0$ for $n < 0$, and $\mathrm{Tor}_0(M, N) \cong M \otimes N$.
(3) $\mathrm{Tor}_n(M, N) \cong \mathrm{Tor}_n(N, M)$.
(4) $Tor_n(M, \_)$ (respectively, $\mathrm{Tor}_n(\_, M)$) is a covariant functor from R-modules to R-modules
(5) If $0 \to M' \to M \to M'' \to 0$ is exact there is a long exact sequence:
$\cdots \to \mathrm{Tor}_n(M', N) \to \mathrm{Tor}_n(M, N) \to \mathrm{Tor}_n(M'', N) \to \mathrm{Tor}_{n-1}(M', N) \to \cdots$
(6) The map induced on $\mathrm{Tor}_n(M, N)$ by multiplication by $r \in R$ on M (or N) is multiplication by r.
(7) The module $\mathrm{Tor}_n(M, N)$ is killed by $\mathrm{Ann}_R M + \mathrm{Ann}_R N$.
(8) If M or N is flat (e.g., if either is free or projective), then $\mathrm{Tor}_n(M, N) = 0$ for $n \geq 1$
(9) If R is Noetherian and M, N, are finitely generated, so is $\mathrm{Tor}_n(M, N)$ for all n.

From our discussion of minimal resolutions we obtain:

Theorem. *Let M be a finitely generated module over a local ring $(R,\, m,\, K)$. The modules $Tor_i(M,\, K)$ are finite-dimensional vector spaces over K, and $\dim_K\big(Tor_i(M,\, K)\big)$ is the same as the rank of the i th free module in a minimal free resolution of M.*

Moreover the following conditions on M are equivalent:

(1) *In a minimal free resolution $P_\bullet$ of M, $P_{n+1} = 0$.*
(2) *The projective dimension of M is at most n.*
(3) *$Tor_{n+1}(M,\, K) = 0$.*
(4) *$Tor_i(M,\, K) = 0$ for all $i \geq n+1$.*

It follows that a minimal free resolution of M is also a shortest possible projective resolution of M. In particular, M has finite projective dimension (respectively, infinite projective dimension) if and only its minimal free resolution is finite (respectively, infinite.)

Proof. If we take a minimal free resolution $P_\bullet$ of M, because the image of every G_j is in mG_{j-1}, when we apply $\_ \otimes_R K$ the maps become 0, while $G_i \otimes K$ is a vector space V_i over K whose dimension is the same as the rank of G_i. Hence, the homology of $P_\bullet \otimes_R K$ at the i th spot is V_i, and the first statement follows. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) (note that once one of the P_j is 0, all the P_k for $k \geq j$ are 0). The last implication follows from the first assertion of the Theorem. It is also clear that (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1). There cannot be a projective resolution shorter than the minimal resolution, for if $P_j \neq 0$ then $\mathrm{Tor}_j(M,\, K) \neq 0$, and if there were a shorter resolution it could be used to compute $\mathrm{Tor}_j(M,\, K)$, which would have to vanish. The final statement is then clear. □

We shall next use some elementary facts about Tor to prove that over a regular local ring, every finitely generated module has finite projective dimension. We first note:

Proposition. *Let R be a ring and let $x \in R$ an element.*

(a) *Given an exact sequence $\mathcal{Q}_\bullet$ of modules*

$$\cdots \to Q_{n+1} \to Q_n \to Q_{n-1} \to \cdots$$

(it may be doubly infinite) such that x is a nonzerodivisor on all of the modules Q_n, the complex $\overline{\mathcal{Q}}_\bullet$ obtained by applying $\_ \otimes R/xR$, which we may alternatively describe as

$$\cdots \to Q_{n+1}/xQ_{n+1} \to Q_n/xQ_n \to Q_{n-1}/xQ_{n-1} \to \cdots,$$

is also exact.

(b) *If x is a nonzerodivisor in R and is also a nonzerodivisor on the module M, while $xN = 0$, then for all i, $Tor_i^R(M,\, N) \cong Tor_i^{R/xR}(M/xM,\, N)$.*

Proof. (a) We get a short exact sequence of complexes $0 \to \mathcal{Q}_\bullet \xrightarrow{x\cdot} \mathcal{Q}_\bullet \to \overline{\mathcal{Q}}_\bullet \to 0$ which, at the n th spots, is $0 \to Q_n \xrightarrow{x\cdot} Q_n \to Q_n/xQ_n \to 0$ (exactness follows because x is a nonzerodivisor on every Q_n). The snake lemma yields that

$$\cdots \to H_n(\mathcal{Q}_\bullet) \to H_n(\mathcal{Q}_\bullet) \to H_n(\overline{\mathcal{Q}}) \to H_{n-1}(\mathcal{Q}_\bullet) \to \cdots$$

is exact, and since $H_n(\mathcal{Q}_\bullet)$ and $H_{n-1}(\mathcal{Q}_\bullet)$ both vanish, so does $H_n(\overline{\mathcal{Q}}_\bullet)$.

(b) Consider a free resolution $(*)\ \cdots \to P_n \to \cdots P_0 \to M \to 0$ for M. By part (a), this remains exact when we apply $\_ \otimes_R R/xR$, which yields a free resolution of M/xM over R/xR. Let $P_\bullet$ be the complex $(*)$ with M replaced by 0. Then $\mathrm{Tor}_n^R(M, N)$ is the homology at the nth spot of $P_\bullet \otimes_R N$. Since x kills N, $(R/xR) \otimes_{R/xR} N \cong N$. Thus, $\mathrm{Tor}_n(M, N)$ is the homology at the nth spot of $\big(P_\bullet \otimes_R (R/xR)\big) \otimes_{R/xR} N$, and since $P_\bullet \otimes_R R/xR$ is a free resolution of M/xM over R/xR, this is also $\mathrm{Tor}_n^{R/xR}(M, N)$. □

We can now prove:

Theorem. *If (R, m, K) is a regular local ring of Krull dimension d, then for every finitely generated R-module M, the projective dimension of M is at most d.*

Proof. We use induction on $\dim(R)$. If $\dim(R) = 0$, then the maximal ideal of R is generated by 0 elements, and is a field, so that every R-module is free and has projective dimension at most 0.

Now suppose $\dim(R) \geq 1$. Let M be a finitely generated R-module. It suffices to prove that $\mathrm{Tor}_n(M, K) = 0$ for $n > d$. We can form a short exact sequence $0 \to M_1 \to P \to M \to 0$ where P is free. Since $M_1 \subseteq P$, if we choose a regular paramter $x \in M$, x is not a zerodivisor on M_1. Hence, $\mathrm{Tor}_n^R(M_1, K) \cong \mathrm{Tor}_n^{R/xR}(M_1/xM_1, K)$ by the preceding Proposition. The long exact sequence for Tor coming from the short exact sequence $0 \to M_1 \to P \to M \to 0$ shows that $\mathrm{Tor}_{n+1}^R(M, K) \cong \mathrm{Tor}_n^R(M_1, K) \cong \mathrm{Tor}_n^{R/xR}(M_1/xM_1, K)$ for $n \geq d$, and the last term vanishes by the induction hypothesis, since R/xR is again regular. □

The converse is much more difficult. We need several preliminary results. We write $\mathrm{pd}_R M$ or, if R is understood from context, $\mathrm{pd}\, M$ for the projective dimension of M over R. We first note:

Theorem. *Let (R, m, K) be a local ring.*

Given a finite exact sequence of finitely generated R-modules such that every term but one has finite projective dimension, then every term has finite projective dimension.

In particular, given a short exact sequence

$$0 \to M_2 \to M_1 \to M_0 \to 0$$

of finitely generated R-modules, if any two have finite projective dimension over R, so does the third. Moreover:

(a) $pd\, M_1 \leq \max\{pd\, M_0,\, pd\, M_2\}$.
(b) *If $pd\, M_1 < pd\, M_0$ are finite, then $pd\, M_2 = pd\, M_0 - 1$. If $pd\, M_1 \geq pd\, M_0$, then $pd\, M_2 \leq pd\, M_1$.*
(c) $pd\, M_0 \leq \max\{pd\, M_1, pd\, M_2 + 1\}$.

Proof. Consider the long exact sequence for Tor:

$$\cdots \to \mathrm{Tor}_{n+1}^R(M_1, K) \to \mathrm{Tor}_{n+1}^R(M_0, K) \to \mathrm{Tor}_n^R(M_2, K)$$
$$\to \mathrm{Tor}_n^R(M_1, K) \to \mathrm{Tor}_n^R(M_0, K) \to \cdots$$

If two of the M_i have finite projective dimension, then two of any three consecutive terms are eventually 0, and this forces the third term to be 0 as well.

The statements in (a), (b), and (c) bounding some $\mathrm{pd}\, M_j$ above for a certain $j \in \{0, 1, 2\}$ all follow by looking at trios of consecutive terms of the long exact sequence such that the middle term is $\mathrm{Tor}_n^R(M_j, K)$. For n larger than the specified upper bound for $\mathrm{pd}_R M_j$, the Tor on either side vanishes. The equality in (b) for the case where $\mathrm{pd}\, M_1 < \mathrm{pd}\, M_0$ follows because with $n = \mathrm{pd}\, M_0 - 1$, $\mathrm{Tor}_{n+1}^R(M_0,\, K)$ injects into $\mathrm{Tor}_n^R(M_2,\, K)$.

The statement about finite exact sequences of arbitrary length now follows by induction on the length. If the length is smaller than three we can still think of it as 3 by using terms that are 0. The case of length three has already been handled. For sequences of length 4 or more, say

$$0 \to M_k \to M_{k-1} \to \cdots \to M_1 \to M_0 \to 0,$$

either M_k and M_{k-1} have finite projective dimension, or M_1 and M_0 do. In the former case we break the sequence up into two sequences

$$0 \to M_k \to M_{k-1} \to B \to 0$$

and

$$(*) \quad 0 \to B \to M_{k-2} \to \cdots \to M_1 \to M_0 \to 0.$$

The short exact sequence shows that $\mathrm{pd}\, B$ is finite, and then we may apply the induction hypothesis to $(*)$. If M_1 and M_0 have finite projective dimension we use exact sequences

$$0 \to Z \to M_1 \to M_0 \to 0$$

and

$$0 \to M_k \to M_{k-1} \to \cdots \to M_2 \to Z \to 0$$

instead. □

Lemma. *If M has finite projective dimension over (R, m, K) local, and $m \in \mathrm{Ass}\,(R)$, then M is free.*

Proof. If not, choose a minimal free resolution of M of length $n \geq 1$ and suppose that the left hand end is

$$0 \to R^b \xrightarrow{A} R^a \to \cdots$$

where A is an $a \times b$ matrix with entries in m. The key point is that the matrix A cannot give an injective map, because if $u \in m - \{0\}$ is such that $\mathrm{Ann}_R u = m$, then A kills a column vector whose only nonzero entry is u. □

Lemma. *If M has finite projective dimension over R, and x is not a zerodivisor on R and not a zerodivisor on M, then M/xM has finite projective dimension over both R and over R/xR.*

Proof. Let $P_\bullet$ be a finite projective resolution of M over R. Then $P_\bullet \otimes_R R/xR$ is a finite complex of projective R/xR-modules whose homology is $\mathrm{Tor}_n^R(M,\, R/xR)$, which is 0 for

$n \geq 1$ when x is not a zerodivisor on R or M. This gives an (R/xR)-projective resolution of M over R/xR. The short exact sequence

$$0 \to P \xrightarrow{x} P \to P/xP \to 0$$

shows that each P/xP has projective dimension at most 1 over R, and then M/xM has finite projective dimension over R by the Proposition above. □

Lemma. *Let (R, m, K) be local, let I_n denote the $n \times n$ identity matrix over R, let x be an element of $m - m^2$, and let A, B be $n \times n$ matrices over R such that $xI_n = AB$. Suppose that every entry of A is in m. Then B is invertible.*

Proof. We use induction on n. If $n = 1$, we have that $(x) = (a)(b) = (ab)$, where $a \in m$. Since $x \notin m^2$, we must have that b is a unit. Now suppose that $n > 1$. If every entry of B is in m, the fact that $xI_n = AB$ implies that $x \in m^2$ again. Thus, some entry of B is a unit. We permute rows and columns of B to place this unit in the upper left hand corner. We multiply the first row of B by its inverse to get a 1 in the upper left hand corner. We next subtract multiples of the first column from the other columns, so that the first row becomes a 1 followed by a string of zeros. We then subtract multiples of the first row from the other rows, so that the first column becomes 1 with a column of zeros below it. Each of these operations has the effect of multiplying on the left or on the right by an invertible $n \times n$ matrix. Thus, we can choose invertible $n \times n$ matrices U and V over R such that $B' = UBV$ has the block form

$$B' = \begin{pmatrix} 1 & 0 \\ 0 & B_0 \end{pmatrix},$$

where the submatrices 1, 0 in in the first row are 1×1 and $1 \times (n-1)$, respectively, while the submatrices 0, B_0 in the second row are $(n-1) \times 1$ and $(n-1) \times (n-1)$, respectively.

Now, with

$$A' = V^{-1}AU^{-1},$$

we have

$$A'B' = V^{-1}AU^{-1}UBV = V^{-1}(AB)V = V^{-1}(xI_n)V = x(V^{-1}I_nV) = xI_n,$$

so that our hypothesis is preserved: A' still has all entries in m, and the invertibility of B has not been changed. Suppose that

$$A' = \begin{pmatrix} a & \rho \\ \gamma & A_0 \end{pmatrix}$$

where $a \in R$ (technically a is a 1×1 matrix over R), ρ is $1 \times (n-1)$, γ is $(n-1) \times 1$, and A_0 is $(n-1) \times (n-1)$. Then

$$xI_n = A'B' = \begin{pmatrix} a(1) + \rho(0) & a(0) + \rho B_0 \\ \gamma(1) + A_0(0) & \gamma(0) + A_0 B_0 \end{pmatrix} = \begin{pmatrix} a & \rho B_0 \\ \gamma & A_0 B_0 \end{pmatrix}$$

from which we can conclude that $xI_{n-1} = A_0 B_0$. By the induction hypothesis, B_0 is invertible, and so B' is invertible, and the invertibility of B follows as well. □

The following is critical in proving that if K has finite projective dimension over (R, m, K) then R is regular.

Theorem. *If M is finitely generated and has finite projective dimension over the local ring $(R,\, m,\, K)$, and $x \in m - m^2$ kills M and is not a zerodivisor in R, then M has finite projective dimension over R/xR.*

Proof. We may assume M is not 0. M cannot be free over R, since $xM = 0$. Thus, we may assume $\mathrm{pd}_R M \geq 1$. We want to reduce to the case where $\mathrm{pd}_R M = 1$. If $\mathrm{pd}_R M > 1$, we can think of M as a module over R/xR and map $(R/xR)^{\oplus h} \twoheadrightarrow M$ for some h. The kernel M_1 is a first module of syzygies of M over R/xR. By part (b) of the second Theorem on p. 5, $\mathrm{pd}_R M_1 = \mathrm{pd}_R M - 1$. Clearly, if M_1 has finite projective dimension over R/xR, so does M. By induction on $\mathrm{pd}_R M$ we have therefore reduced to the case where $\mathrm{pd}_R M = 1$. To finish the proof, we shall show that if $x \in m - m^2$ is not a zerodivisor in R, $xM = 0$, and $\mathrm{pd}_R M = 1$, then M is free over R/xR.

Consider a minimal free resolution of M over R, which will have the form

$$0 \to R^n \xrightarrow{A} R^k \to M \to 0$$

where A is an $k \times n$ matrix with entries in m. If we localize at x, we have $M_x = 0$, and so

$$0 \to R_x^n \to R_x^k \to 0$$

is exact. Thus, $k = n$, and A is $n \times n$. Let e_j denote the j th column of the identity matrix I_n. Since $xM = 0$, every xe_j is in the image of A, and so we can write $xe_j = Ab_j$ for a certain $n \times 1$ column matrix b_j over R. Let B denote the $n \times n$ matrix over R whose columns are $b_1, \, \ldots \, , b_n$. Then $xI_n = AB$. By the preceding Lemma, B is invertible, and so A and $AB = xI_n$ have the same cokernel, up to isomorphism. But the cokernel of xI_n is $(R/xR)^{\oplus n} \cong M = \mathrm{Coker}\,(A)$, as required. $\square$

We can now prove the result that we are aiming for, which completes the proof of the Theorem stated at the end of the previous lecture.

Theorem. *Let (R, m, K) be a local ring such that $pd_R K$ is finite. Then R is regular.*

Proof. If $m \in \mathrm{Ass}\,(R)$, then we find that K is free. But $K \cong R^n$ implies that $n = 1$ and R is a field, as required. We use induction on $\dim\,(R)$. The case where $\dim\,(R) = 0$ follows, since in that case $m \in \mathrm{Ass}\,(R)$.

Now suppose that $\dim\,(R) \geq 1$ and $m \notin \mathrm{Ass}\,(R)$. Then m is not contained in m^2 nor any of the primes in $\mathrm{Ass}\,(R)$, and so we can choose $x \in m$ not in m^2 nor in any associated prime. This means that x is not a zerodivisor in R. By the preceding Theorem, the fact that K has finite projective dimension over R implies that it has finite projective dimension over R/xR. By the induction hypothesis, R/xR is regular. Since $x \notin m^2$ and x is not a zerodivisor, both the least number of generators of the maximal ideal and the Krull dimension drop by one when we pass from R to R/xR. Since R/xR is regular, so is R. $\square$

We have finially proved the result we were aiming for, and we have now completed the argument given much earlier that a localization of a regular local ring is regular. We also note:

Theorem. *Let $R \to S$ be a faithfully flat homomorphism of Noetherian rings. If S is regular, then R is regular.*

Proof. Let P be a maximal ideal of R. Then $PS \neq S$, and there is a maximal ideal Q of S lying over P. It suffices to show that every R_P is regular, and we have that $R_P \to S_Q$ is flat and local. Thus, we have reduced to the case where R and S is local and the map is local. Take a minimal free resolution of R/P over R. If R is not regular, this resolution is infinite. Apply $S \otimes_R \_$. Since S is R-flat, we get a free resolution of S/PS over S. Since P maps into Q, this resolution is still minimal. Thus, S/PS has infinite projective dimension over S, contradicting the fact that S is regular. $\square$