Math 615, Winter 2011 **Problem Set #1**
Due: January 28.

1. Let N be a module over the Noetherian ring R. Let P be a prime ideal of R. Let $K = R_P/PR_P \cong \mathrm{frac}(R/P)$. Prove that the number of copies of $E_R(R/P)$ occurring in any decomposition of $E_R(N)$ as a direct sum of injective hulls of prime cyclic modules (i.e., cyclic modules R/Q with prime annihilator Q) is $\dim_K \mathrm{Hom}_{R_P}(K,\, N_P)$.

2. Prove that if S is an R-flat algebra, an injective module E over S is injective over R. In particular, this holds when S is a localization of R, and the fraction field of any domain D is an injective module over D.

3. Let $(V,\, m,\, K)$ be a Noetherian valuation domain, with maximal ideal $m = tV \neq (0)$. Let $\mathcal{F}$ denote the fraction field of V. Let $E = \mathcal{F}/V$. Prove that the annihilator of m in E is the one-dimensional K-vector space spanned by the image of $1/t$, and that E is an injective hull for $K = V/m$ over V.

4. Let $(R,\, m,\, K)$ be a regular local ring of Krull dimension 2 (so that m is generated by two elements). You may assume that R is a UFD, which follows from a theorem. Let $m = (x, y)$. Prove that an R-module E is injective if and only if it is divisible and for any two elements $u,\, v \in E$ such that $yu = xv$, there exists an element $w \in E$ such that $u = xw$ and $v = yw$.

5. Let $x_1,\, \ldots\, , x_n,\, \ldots$ be a countably infinite sequence of indeterminates over the field K, let $T = K[x_1,\, \ldots\, , x_n,\, \ldots]$, let $\mathcal{M}$ be the maximal ideal generated by the x_n, and let $R = T/\mathcal{M}^2$. Then R has a K-vector space basis consisting of 1 and the images $\overline{x}_n$ of the x_n. Let $m = \mathcal{M}/\mathcal{M}^2$ be the maximal ideal of R. Let E_n be an injective hull for $K\overline{x}_n \cong R/m$ over R, that is, every $E_n \cong E_R(R/m)$. As an R-module $m = \bigoplus_{n=1}^{\infty} R\overline{x}_n \cong \bigoplus_n K\overline{x}_n$, and the direct sum of the injections $K\overline{x}_n \hookrightarrow E_n$ yields an injection $\theta : m \hookrightarrow \bigoplus_{n=1}^{\infty} E_n$. Does θ extend to R? Is $\bigoplus_{n=1}^{\infty} E_n$ an injective R-module?

6. If R is a ring and M an R-module, we can give the R-module $R \oplus M$ the structure of am extension ring of R by defining $(r \oplus u)(s \oplus v) = (rs) \oplus (rv + su)$, so that in this ring M is an ideal and $M^2 = 0$. You may assume this.

Let notation be as in problem **3.** above. Show that the ring $S = V \oplus E$, where $E = E_V(K)$, is an essential extension, as an S-module, of $K \subseteq E_V(K)$. Note that S is one-dimensional, quasilocal with maximal ideal $m \oplus E$ and unique minimal prime E. Show that this extension $K \hookrightarrow S$ is no longer essential if one localizes at t, or at the prime ideal E in S.

EXTRA CREDIT 1. Let R and S be finitely generated algebras over a field K. Let E be an injective R-module and let F be an injective S-module. Is $E \otimes_K F$ necessarily injective as an $(R \otimes_K S)$-module?