Math 615, Winter 2011 **Problem Set #1 : Solutions**

1. The number of copies of $E_R(R/P)$ in the direct sum decomposition does not change when we localize at P, since $E_{R_P}(N_P) \cong E_R(N)_P$. It follows that we may assume that $(R,\, P)$ is local. Then for any R-module M, $\mathrm{Hom}_R(K, M) \cong \mathrm{Ann}_M P$. Quite generally, if $N \subseteq M$ is essential, then $\mathrm{Ann}_N I \subseteq \mathrm{Ann}_M I$ for any ideal I of R: every nonzero element of $\mathrm{Ann}_I M$ has a nonzero multiple in N, and this multiplle is clearly killed by I and so in $\mathrm{Ann}_N I$. Hence, $\mathrm{Hom}_R(K,\, N) = Ann_N P \subseteq Ann_{E_R(N)} P = \mathrm{Hom}_{R_P}(K,\, E_R(N))$ is essential. Since this is an inclusion of K-vector spaces, for it to be essential we must have $\mathrm{Hom}_R(K,\, N) = \mathrm{Hom}_{R_P}(K,\, E_R(N))$, since an inclusion of vectors spaces always splits. The K-vector space dimension of the latter is the number of copies of $E_R(K)$ in a direct sum decomposition for $E_R(N)$ by a class theorem, and this is equal to the K-vector space dmension of the former, as required. $\square$

2. By the universal mapping property for base change, the functor $\mathrm{Hom}_R(\_\,,\, E)$ is isomorphic to the functor $\mathrm{Hom}_S(S \otimes_R \_\,,\, E)$. (This is true for any S-module E viewed as an R-module by restriction of scalars.) This is the composition of the functors $S \otimes_R \_\,$, which is exact because S is flat over R, and $\mathrm{Hom}_S(\_\,,\, E)$, which is exact because E is injective over S. It follows that the functor $\mathrm{Hom}_R(\_\,,\, E)$ is exact and, hence, that E is injective over R. $\square$

3. Note that a discrete valuation ring $(V,\, tV,\, K)$ is a PID, since every nonzero element is a unit times a power t^k of t, with $t \in \mathbb{N}$. To show that $\mathcal{F}/V$ is injective, it suffices to show that it is a divisible V-module, which is clear because it is a homomorphic image of the divisible V-module $\mathcal{F}$. Every element of $\mathcal{F} - \{0\}$ is a unit of V times a power t^k of t, with $k \in \mathbb{Z}$. Thus, the nonzero elements of $\mathcal{F}/V$ can be represented by an element of the form ut^{-k}, where u is a unit of V and $k > 0$. If we multiply by $u^{-1}t^{k-1}$, we obtain t^{-1}, which is killed by t and so by $tV = m$. Thus, the cyclic module generated over V by the class of t^{-1} is $\cong K$, and the extension is essential. Thus, $\mathcal{F}/V \cong E_V(K)$. $\square$

4. By a class result, E is injective if and only if maps from prime ideals of R to E extend to R. [Consider a map to E defined on $N \subseteq M$ which supposedly cannot be extended further. Pick $m \in M - N$. One may replace m by a multiple such that the ideal multiplying m into N is prime. (The image of m in M/N has a multiple which has prime annihilator.)] This is trivial for the 0 prime ideal, and extending maps from principal ideals of a domain can be done if and only if the module is divisible. Hence, we may assume that E is divisibile. The only remaining prime ideal of R is the maximal ideal $(x,\, y)R$. Thus, a divisible module E is injective if every map from $(x,\, y)R$ to E extends to a map $R \to E$. We can map $R^2 \twoheadrightarrow m$ by $(r, s) \mapsto rx + sy$. If (r, s) is in the kernel, then $rx = -sy$. Since x is prime and does not divide y, it must divide s, say $s = ax$. But then it follows that $r = -ay$, and $(r, s) = a(-y, x)$. Thus, $m \cong R^2/R(-y, x)$. To give a map $m \to E$ is therefore the same as to give a map $R^2 \to E$ that kills $(-y, x)$, i.e., to give $u, v \in E$ such that $-yu + xv = 0$, where $u,\, v \in E$ are to be the images of $x,\, y \in m$. The problem of extending the map to R is the same as the problem of specifying the value w for 1 so that x will map to u and y to v, i.e., such that $xw = u$ and $yw = v$. $\square$

5. The map cannot be extended to R because the image of 1 in the direct sum of the E_n will be nonzero in only finitely many E_n, which implies that the image of the map is contained in only finitely many E_n. But each x_j maps to a nonzero element in the corresponding E_j, a contradiction. $\square$

6. Consdier a nonzero element $v \oplus e$ in $S = V \oplus E$. If v is not 0 it has the form ut^k where u is a unit of V and $k \geq 0$. In this case we may take $(0 \oplus u^{-1}t^{-k+1})(ut^k \oplus e)$ to obtain $0 \oplus t^{-1} \in 0 \oplus K$. If the nonzero element has the form $0 \oplus e$ where $e \neq 0$, we may multiply by an element of $V \oplus 0$ to get a nonzero element of K, since $K \hookrightarrow E$ is essential. Thus, $K = 0 \oplus K \hookrightarrow V \oplus E$ is essential. Since every element of E is killed by a power of t, if we localize at t, E becomes 0, while $V_t \cong \mathcal{F}$. Thus, $S_t \cong \mathcal{F}$ is a field. The extension is no longer essential, since the submodule becomes 0 while the ambient module does not. Note that since S_t is a field, it has a unique maximal ideal, which means that S_t is also the localization of S at the prime ideal $0 \oplus E$.

Extra Credit 1. Let $K[X]$ and $K[Y]$ be polynomial rings in one variable over the field K. $K(X)$ (resp., $K(Y)$) is divisible as a module over the principal ideal domain $K[X]$ (resp., $K[Y]$), and, hence, injective. But $M = K(X) \otimes_K K(Y)$ is not injective over $K[X] \otimes_K K[Y] \cong K[X, Y]$: one can see this because, for example, $X + Y$ has no inverse, and so M is not divisible. $\square$