Math 615, Winter 2011 **Problem Set #2**
Due: Monday, Febraury 14

1. Let R be any ring, let J be a finitely generated ideal of R, and let E be any injective R-module. Let $M^\vee$ denote $\mathrm{Hom}_R(M, E)$. Prove that $(\mathrm{Ann}_M J)^\vee \cong M^\vee/JM^\vee$.

2. Let (R, m, K) be a local ring, and let I be an m-primary ideal such that the socle in R/I is a one-dimensional K-vector space. Suppose that $I \subseteq J$. Prove or disprove that $I : (I : J) = J$, where $I : J = \{r \in R : rJ \subseteq I\}$.

3. Let $I \subseteq J$ be ideals of the Noetherian ring R. Show that there is a natural map $H^i_J(M) \to H^i_I(M)$ for every integer i using the definition of local cohomology as a direct limit of Ext modules. Show that the map you define is an isomorphism if I and J have the same radical.

4. Let M be a finitely generated module over a Noetherian ring R, let I be an ideal of R with $IM \neq M$, and let $x_1, \ldots, x_d \in I$ be a maximal regular sequence on M.

(a) Part of the long exact sequence for local cohomology coming from the short exact sequence $0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0$ gives

$$0 \to H^{d-1}_I(M/x_1M) \xrightarrow{\partial} H^d_I(M) \xrightarrow{x_1} H^d_I(M)$$

Show that ∂ is an essential extension, and, hence, that $\mathrm{Ass}\,(H^d_I(M)) = \mathrm{Ass}\,(H^{d-1}_I(M/x_1M))$.

(b) Conclude that $\mathrm{Ass}\,(H^d_I(M)) = \mathrm{Ass}\,(H^0_I(M/(x_1, \ldots, x_d)M))$. Hence, $\mathrm{Ass}\,(H^d_I(M))$ is finite.

5. Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let $d_1, \ldots, d_n > 0$ be integers. Let I be the ideal of T generated by the products x_ix_j for $i \neq j$ and the elements $x_i^{d_i}$, $1 \leq i \leq n$. Let $R = T/I$. Let m be the maximal ideal $(x_1, \ldots, x_n)R$. Determine the length of $E = E_R(R/m)$, and determine the least number of generators of E.

6. Let R be a Noetherian ring, let I be an ideal and let M be a module of finite length. Let $f \in R$ be any element and let $J = I + fR$. Show that there is an exact sequence

$$0 \to H^0_J(M) \to H^0_I(M) \to H^0_I(M_f) \to 0$$

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EXTRA CREDIT 2. Let (R, m, K) be a local ring and suppose that we have a map $K \hookrightarrow R$ such that the composite $K \to R \twoheadrightarrow K$ is an isomorphism. (We then say that K is a *coefficient field* for R. (This holds, for example, whenever R is complete and contains any field.) Let E denote the set of K-linear maps $R \to K$ that kill m^n for some integer $n > 0$. Prove that E is an injective hull for K over R.

EXTRA CREDIT 3. Let $K[u, v, x, y]$ be a polynomial ring over a field, and let R be the subring $K[xu, xv, yu, yv]$. Let $m = (xu, xv, yu, yv)R$. Let $I = (xu, xv)$. Show that every element of $H^2_I(R)$ is killed by a power of m. Does $H^2_I(R)$ have DCC?