Math 615, Winter 2011 **Problem Set #2: Solutions**

**1.** Let $J = (f_1, \dots, f_n)R$, and map $g : M \to M^{\oplus n}$ by $m \mapsto (f_1 m, \dots, f_n m)$. Then $0 \to \mathrm{Ann}_M J \to M \xrightarrow{g} M^{\oplus n}$ is exact. Since $\_^{\vee}$ is an exact, we get an exact sequence $M^{\vee \oplus n} \xrightarrow{g^{\vee}} M^{\vee} \to (\mathrm{Ann}_M J)^{\vee} \to 0$. Since $g^{\vee}(u_1 \oplus \cdots \oplus u_n) = f_1 u_1 + \cdots f_n u_n$ the image of $g^{\vee}$ is $J(M^{\vee})$ and so we have that $M^{\vee}/J(M^{\vee}) \cong (\mathrm{Ann}_M J)^{\vee}$. $\square$

**2.** Replace $R$ by $R/I$ and $J$ by $J/I$. Then $R$ is an Artin local ring such that $E_R(K) = R$, $J$ is an ideal of $R$, and the assertion becomes that the $O :_R (0 :_R J) = J$ (or $\mathrm{Ann}_R(\mathrm{Ann}_R(J)) = J$.) This is true. It was proved in class that if $R$ is complete local and $E = E_R(K)$, then there is a bijection between ideals of $R$ and submodules of $E$: the map in each direction is taking the annihilator (either of the ideal in $E$ or of the submodule in $R$). This is the special case where $E = R$. $\square$

**3.** Since $I \subseteq J$, we have $I^t \subseteq J^t$ for all $t$, giving surjections $R/J^t \twoheadrightarrow R/I^t$. The diagrams

$$\begin{array}{ccc} R/I^{t+1} & \longrightarrow & R/J^{t+1} \\ \downarrow & & \downarrow \\ R/I^t & \longrightarrow & R/J^t \end{array}$$

commute, and hence, we have that for all $i$ and $t$ the diagrams

$$\begin{array}{ccc} \mathrm{Ext}^i_R(R/I^{t+1}, M) & \longleftarrow & \mathrm{Ext}^i_R(R/J^{t+1}, M) \\ \uparrow & & \uparrow \\ \mathrm{Ext}^i_R(R/I^t, M) & \longleftarrow & \mathrm{Ext}^i_R(R/J^t, M) \end{array}$$

commute. Hence, there is an induced map $\varinjlim_t \mathrm{Ext}^i_R(R/J^t, M) \to \varinjlim_t \mathrm{Ext}^i(R/I^t, M)$ i.e., of $H^i_J(M) \to H^i_I(M)$ for all $i$. From this construction it is also easy to see that if $I \subseteq J \subseteq J'$, the map $H^i_{J'}(M) \to H^i_I(M)$ is the composition of the map $H^i_{J'}(M) \to H^i_J(M)$ with the map $H^i_J(M) \to H^i_I(M)$. If $I$ and $J$ have the same radical, then there is an integer $s > 0$ such that $J^s \subseteq I \subseteq J$, and then $I^s \subseteq J^s \subseteq I \subseteq J$. The fact that the map $H^i_J(M) \to H^i_{J^s}(M)$ is an isomorphism follows from the fact that $H^i_J(M)$ is also the direct limit of the cofinal elements $\mathrm{Ext}^i_R(R/J^{st}, M)$ in the direct limit system of $\mathrm{Ext}^i_R(R/J^t, M)$. Similarly, $H^i_I(M) \cong H^i_{I^s}(M)$. We have $H^i_J(M) \xrightarrow{f} H^i_I(M) \xrightarrow{g} H^i_{J^s}(M) \xrightarrow{h} H^i_{I^s}(M)$, where $hg$ is an isomorphism, so that $g$ is injective, and $gf$ is an isomorphism, so that $g$ is onto. Thus, $g$ is an isomorphism, and, hence, so is $g^{-1}(gf) = f$. $\square$

**4.** Let $u \in H^d_I(M)$ be nonzero. Since some power of $I$ kills $u$ and $x_1 \in I$, there is a least positive integer $t$ such that $x_1^t u = 0$. It follows that $x_1^{t-1}u$ is a nonzero multiple of $u$ in the kernel of the map given by multiplcation by $x_1$, and therefore in the image of $\partial$. This shows that $H^{d-1}_I(M/x_1M) \xrightarrow{\partial} H^d_I(M)$ is essential. If $u \in H^d_I(M)$ has prime annihilator $P$, $Ru \cong R/P$ then meets the image of $H^{d-1}_I(M/x_1M)$ in a nonzero element $v$. We may think of $v$ as a nonzero element of $R/P$. Thus, $\mathrm{Ann}_R v = P$. This proves the assertion about the associated primes, and completes the argument for part (b). It then follows by induction on $k$ that $\mathrm{Ass}\,(H_I^d(M)) = \mathrm{Ass}\,(H_I^{d-k}(M/(x_1, \ldots, x_k)M))$ for $0 \leq k \leq d$. Finally, note that $H_I^0(M/(x_1, \ldots, x_d)M))$ is a submodule of the Noetherian module $M/(x_1, \ldots, x_d)M$, so that the set of associated primes is finite. $\square$

**5.** $R$ has $K$-basis consisting of 1 and the elements $x_i^k$ for $1 \leq i \leq n$ and $1 \leq k \leq d_i - 1$. The number of such elements is $1 + \sum_{i=1}^n (d_i - 1) = \sum_{i=1}^n d_i - (n-1)$, and this is both the $K$-vector space dimension of $R$ and the length of $R$ as an $R$-module. By a class theorem, the length of $E_R(K)$ is the same as the length of $R$ for an Artin local ring $R$, and so $E_R(K)$ also has this length. By, a class theorem, the least number $\nu$ of generators of $E$ is the dimension of the socle in $R$. If all of the $d_i = 1$, then $R = K = E$ and $\nu = 1$. In all other cases, the socle in $R$ is spanned as a $K$-vector space by those $x^{d_i - 1}$ such that $d_i \geq 2$, and $\nu$ is the number of values of $i$ such that $d_i \geq 2$. $\square$

**6.** Clearly, $H_J^0(M) \hookrightarrow H_I^0(M)$ by definition ($J^t u = 0 \Rightarrow I^t u = 0$). $\mathrm{Ker}\,(H_I^0(M) \to H_I^0(M_f))$ (the latter is $\cong H_I^0(M)_f$) consists of all $v \in H_I^0(M)$ such that $v$ is killed by a power of $f$ (as well as a power of $I$). Thus, $0 \to H_J^0(M) \to H_I^0(M) \to H_I^0(M_f)$ is exact for all $M$. If $M$ has finite length, $M \cong M_1 \oplus \cdots \oplus M_k$ where each $M_k$ is kiled by a power of a maximal ideal $m_k$. Hence, we may assume that $M$ is killed by a power of a maximal ideal $m$, and we need only show that $\theta : H_I^0(M) \to H_I^0(M_f)$ is onto. But if $f \in m$, then $M_f = 0$, while if $f \notin m$, then $M \cong M_f$ and $\theta$ is an isomorphism.

**EXTRA CREDIT 2.** By a class result, if $(R, m, K) \to (S, n, L)$ is local and module-finite, then $\mathrm{Hom}_R(S, E_R(K))$ is an injective hull for $L$ over $S$. Thus, if $R$ is Artin local and contains $K \subseteq R$ that maps isomorphically onto $R/m$, then $\mathrm{Hom}_K(R, K) \cong E_R(K)$. Thus, the linear functionals $E_n$ from $R \to K$ that kill $m^n$ form an injective hull for $K$ over $R/m^n$, and $E = \bigcup_{n=1}^\infty E_n$. Each $E_n$ is an essential extension of $K = \mathrm{Hom}_K(R/m, K) = E_1$, Since every element of $E$ is in some $E_n$, it follows that $E$ is an essential extension of $K = E_1$. Hence, $E \subseteq E'$ where $E'$ is maximal essential extension of $E$ and, hence, of $K$. Since every element of $E'$ is killed by $m^n$ for some $n$. we can show that $E = E'$ by showing that $\mathrm{Ann}_E m^n \subseteq \mathrm{Ann}_{E'} m^n$ is an isomorphism. The former contains $E_n$, which is an injective hull for $K$ over $R/m^n$, and the latter is an injective hull for $K$ over $R/m^n$. Hence, their lengths are the same, and the inclusion cannot be proper. $\square$

**EXTRA CREDIT 3.** Let $H = H_I^2(R) \cong R_{xuxv}/(R_{xu} + R_{xv})$. $R$ is spanned as a $K$-vector space by all monomials $u^a v^b x^c y^d$ such that $a + b = c + d$ with $a, b, c, d \in \mathbb{N}$ and so $R_{xuxv}$ is spanned by the set $\mathcal{M}$ of monomials such that $a+b = c+d$, $a, b, c \in \mathbb{Z}$ and $d \in \mathbb{N}$. $R_{xu}$ is spanned by the monomials in $\mathcal{M}$ such that $b, d \in \mathbb{N}$ and $R_{xv}$ by those such that $a, d \in \mathbb{N}$. Thus, the quotient $H$ is spanned over $K$ by the set $\mathcal{N}$ of all monomials for which $a+b = c+d$, $a$, $b < 0$, $d \in \mathbb{N}$, and $a+b = c+d$. Any monomial in $\mathcal{N}$, viewed in $H$, is killed by powers of $xu$ and $yu$ (resp., $xv$ and $yv$), since $a$ (resp., $b$) will become nonnegative after multiplication by a large power. This shows that every element of $H$ is killed by a power of $m$. However, $H$ does not have DCC: the issue is whether the annihilator of $m$ in $H$ is a finite-dimensional vector space or not. In fact, the images of the infinitely many elements $u^{-1}v^{-1}x^{-d-2}y^d$ for $d \in \mathbb{N}$ are part of the basis for $H$ and are killed by $m$. $\square$