Math 615, Winter 2011 **Problem Set #3**
Due: Wednesday, March 16

1. Let I and J be ideals of a local ring $(R,\, m,\, K)$ such that the depth of R/I on m and the depth of R/J on m are both d, while the depth of $R/(I+J)$ on m is $d-h$, where $1 \leq h \leq d$. Determine the depth of $R/(I \cap J)$ on m. You may assume that there is an exact sequence $0 \to R/(I\cap J) \to R/I \oplus R/J \to R/(I+J) \to 0$.

2. Let S be a regular local ring with $\dim(S) = n$. You may assume the theorem that all localizations of S are regular. Let M be a finitely generated S-module. Show that a prime P of height h contains the annihilator of $H^i_m(M)$ if and only if $H^{i-(n-h)}_{PR_P}(M_P) \neq 0$. (Make use of local duality both over S and over S_P.)

3. Let $(R,\, m\, K)$ be a homomorphic image of a regular local ring. Suppose that for every minimal prime P_0 of R, the dimension of $R/P_0 = d$, the dimension of R. Suppose also that R_P is Cohen-Macaulay for every prime P except, possibly, m. Prove that $H^i_m(R)$ has finite length for $i < d$.

4. Let K be a field and let $S = K[[x_1,\, \ldots\,, x_n,\, y_1,\, \ldots\,, y_n]]$ be a formal power series ring in $2n$ variables over K, where $n \geq 2$. Let $I = (x_1,\, \ldots\,, x_n)S$ and $J = (y_1,\, \ldots\,, y_n)S$. Let $R = S/(I \cap J)$. Determine explicitly the local cohomology modules $H^i_m(R)$ for $i < n$.

5. Let $\theta : R \hookrightarrow S$ be a local map of local rings such that S is module-finite over R and θ splits as a map of R-modules. Suppose that S is Cohen-Macaulay ring. Prove that R is a Cohen-Macaulay ring.

6. Let $X,\, Y,\, Z, u, v, a, b, c, d$ be power series indeterminates over the field K, where the characteristic of K is not 3, and let $A = K[[X,\, Y,\, Z]]/(X^3+Y^3+Z^3) = K[[x, y, z]]$. Let $S = K[[xu,\, yu,\, zu,\, xv,\, yv,\, zv]] \subseteq A[[u, v]]$. Show that S is module-finite over $R = K[[xu,\, xv,\, yu,\, yv]]$, and that $xu, yv, xv - yu$ is a system of parameters for R and for S. Prove that S is not Cohen-Macaulay. You may assume that $R \cong K[[a, b, c, d]]/(ad - bc)$. (One can show that A is a normal Gorenstein local domain, and this is also true for $A[[u, v]]$. One can also show that S is a direct summand of $A[[u, v]]$ as an S-module, and so it is a normal ring and a direct summand of a Gorenstein ring, but it is not Cohen-Macaulay.)

EXTRA CREDIT 4. Let K be an algebraically closed field, let R and S be finitely generated K-algebras, and let $m \subseteq R$ and $n \subseteq S$ be maximal ideals. Note that $R/m \cong S/n \cong K$ as K-algebras. Then $Q = m \otimes_K S + R \otimes_K n$ is a maximal ideal of $T = R \otimes_K S$. Prove or disprove that $E_T(T/Q) \cong E_R(R/m) \otimes_K E_S(S/n)$.