Math 615, Winter 2011 **Problem Set #3: Solutions**

1. The exact sequence $0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I+J) \to 0$ yields that $H_m^{i-1}(R/I) \oplus H_m^{i-1}(R/J) \to H_m^{i-1}(R/(I+J)) \to H_m^i(R/(I \cap J)) \to H_m^i(R/I) \oplus H_m^i(R/J)$ is exact. If $i \le d-h$, the second and last terms are 0, and, hence, so is the third term. If $i = d-h+1$, the second term is nonzero and injects into the third term. Hence, the depth is always $d-h+1$. $\square$

2. $H_m^i(M)$ is the Matlis dual of $\mathrm{Ext}_S^{n-i}(M,\, S)$ by local duality over S. If we complete they have the same annihilator J in $\widehat{S}$, and if I is the annihilator of $\mathrm{Ext}_S^{n-i}(M,\, S)$ in S, we have that $J = I\widehat{S}$, and $J \cap S = I$. Thus, both have the same annihilator $I = J \cap S$ in S, and $I \subseteq P$ iff $(\mathrm{Ext}_S^{n-i}(M,\, S))_P \neq 0$, or, eqivalently, $N = \mathrm{Ext}_{S_P}^{n-i}(M_P,\, S_P) \neq 0$. By local duality over S_P, the Matlis dual of N over S_P, which is nonzero iff N is nonzero, is $H_{PR_P}^{h-(n-i)}(M_P)$ and $h-(n-i) = i-(n-h)$. $\square$

3. Let $M = R = S/\mathfrak{A}$ with notation as in **#2**. The issue is then whether the Matlis dual $\mathrm{Ext}^{n-i}P_S(R,\, S)$ has finite length, and this is equivalent to whether a prime $P \neq m$ of S can contain the annihilator. It will suffice to show this does not happen. Clearly, we must have $P \supseteq \mathfrak{A}$, and so P contains a minimal prime of $\mathfrak{A}$, which will correspond to a minimal prime of R. Then $R_P \cong (S/\mathfrak{A})_P$ is Cohen-Macaulay of dimension δ between 0 and $d-1$, and the height h of P in S is δ plus the height of $\mathfrak{A}$, or $\delta + n - d$. By Problem **#2.**, P contains the annihilator if and only if $H_{PR_P}^{h-n+i}(R_P) \neq 0$. But $h - n + i = \delta - d + i < \delta$, a contradiction, because R_P is Cohen-Macaulay and so $H_{PR_P}^{j}(R_P) \neq 0$ if and only if $j = \delta$.

4. From the long exact sequence in **#1.**, $H_m^i(R/(I \cap J)) \cong H_m^{i-1}(R/(I+J))$ for $i < n$, since $H^j(R/I) = H^j(R/J) = 0$ for $j < n$. But $R/(I+J) = K$. Hence $H_m^i(R/(I \cap J)) \cong K$ if $i = 1 < n$, and is 0 for other values of $i < n$.

5. Since S is a module-finite extension of R, we have that $\dim(S) = \dim(R)$. Let $x_1, \ldots, x_n$ be a system of parameters in R and let $I = (x_1, \ldots, x_n)R$. Then R/I has dimension 0, and $R/I \to S/IS$ is module-finite, which implies that $\dim(S/IS) = 0$. Hence, $x_1, \ldots, x_n$ is also a system of parameters for S, and it follows that it is a regular sequence on S. We know that $S = R \oplus W$ over R. It will suffice to show that $x_1, \ldots, x_n$ is a possibly improper regular sequence on R, since we know $I \subseteq m \neq R$. But, in complete generality, $x_1, \ldots, x_n$ is a possibly improper regular sequence on $V \oplus W$ iff $x_1, \ldots, x_n$ is possibly improper regular sequence on both V and W. If $n = 1$ this is clear, and the general case follows by a straightforward induction. $\square$

6. Since $x^3 + y^3 + z^3 = 0$, we may multiply by u^3 to see that $(uz)^3 \in R$, and $(vz)^3 \in R$ similarly. The domain R is the completion at the homogeneous maximal ideal of $R_0 = K[xu,\, xv,\, yu,\, yv]$, and R_0 has dimension 3 because $xu,\, xv,\, yu$ are algebraically independent (even if we specialize x to 1) and $yv = (xv)(yu)/xu$. We can map the polynomial ring $K[a,b,c,d]$ onto R_0 as a K-algebra by sending a,b,c,d to $xu,\, xv,\, yu,\, yv$ resp., and the kernel must be a height one prime and, hence, generated by the irreducible $ad - bc$, which is in the kernel. Thus, $R_0 \cong K[a,b,c,d]/(ad-bc)$, and we may complete to obtain $R \cong K[[a,b,c,d]]/(ad-bc)$. Killing $a = xu$, $d = yv$ and $b - c = xv - yu$ yields $K[[b,c]]/(b-c, -bc) \cong K[[b]]/(b^2)$, which has dimension 0. Hence, $xu,\, yv,\, xv - yu$ is a system of parameters for R and, as in **#5.**, for S. Finally, to see that S is not Cohen-Macaulay, note that $(zu)(zv)(xv-yu) = (zv)^2xu - (zu)^2yv$, but that $(zu)(zv) \notin (xu, yv)$. One may see this last fact as follows: if we work in $S_0 = K[xu,\, yu,\, zu,\, xv,\, yv,\, zv]$ this is true: in fact, it is true even if we specialize u, v to 1. If we localize at the homogeneous maximal m ideal of S_0 it remains true, because all zerodivisors on (xu, yv) are in one of its associated primes, which is homogeneous, and so in m. Because completion is faithfully flat, it remains true when we complete. $\square$

EXTRA CREDIT 4. Map polynomial rings in finitely many variables T, U onto R, S, respectively, and let $\mathcal{M}, \mathcal{N}$ be the inverse images of m, n resp. Since K is algebraically closed, we may change the variables by subtracting scalars and assume that $T = K[x_1, \ldots, x_r]$ and $\mathcal{M} = (x_1, \ldots, x_r)$ while $U = K[y_1, \ldots, y_s]$ and $\mathcal{N} = (y_1, \ldots, y_s)$. Then $R = T/I$ with $I \subseteq \mathcal{M}$ and $S = U/J$ with $J \subseteq \mathcal{N}$. Let $V = T \otimes_K U$. In this case it is easy to check $E_V(V/(\mathcal{M}, \mathcal{N})) \cong E_T(T/\mathcal{M}) \otimes_K E_U(U/\mathcal{N})$ from our description in class of these injective hulls in the polynomial ring case: one gets the localization of the ring at the product of the variables mod the sum of the localizations at the various products of all the variables but one in each of the three cases. Alternatively one has that every monomial that is strictly negative in all of the x_i and y_j is uniquely the product of such a monomial in the x_i and such a monomial in the y_j. When $I = (f_1, \ldots, f_k)$ is a finitely generated ideal of T, G is a T-module, and H is a U-module, $\mathrm{Ann}_{G\otimes_K H} I \cong \mathrm{Ann}_G I \otimes_K H$. This follows from applying the fact that $\__K \otimes N$ preserves exactness and the exact sequence $0 \to G \xrightarrow{\alpha} G^k$ where $\alpha(g) = (f_1 g, \ldots, f_k g)$. Then $E_{R\otimes_K S}(K) \cong E_{V/(I,J)}(K) \cong \mathrm{Ann}_J(\mathrm{Ann}_I(E_V(K)) \cong \mathrm{Ann}_J(\mathrm{Ann}_I(E_T(K) \otimes_K E_U(K)) \cong \mathrm{Ann}_J((\mathrm{Ann}_I E_T(K)) \otimes_K E_U(K)) \cong$ (similarly) $\mathrm{Ann}_I E_T(K)) \otimes_K \mathrm{Ann}_J E_U(K)$ which is $E_R(K) \otimes_K E_S(K)$. $\square$